Intro Video: Integrals resulting in inverse trigonometric functions

Math F251X Calculus 1

Recall

$$\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

$$\frac{d}{dx}\left(\arctan(x)\right) = \frac{1}{1+x^2} \Rightarrow \int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

<u>Example</u>: $\displaystyle\int \frac{1}{1+16x^2}\,dx = \int \frac{1}{1+(4x)^2}\,dx$

$u = 4x$

$\frac{du}{4} = dx$

$$= \frac{1}{4}\int \frac{1}{1+u^2}\,du = \frac{1}{4}\arctan(u) + C$$

$$= \frac{1}{4}\arctan(4x) + C$$

Check: $\frac{d}{dx}\left(\frac{1}{4}\arctan(4x) + C\right)$

$$= \frac{1}{4}\left(\frac{1}{1+(4x)^2}\right)(4) = \frac{1}{1+16x^2}$$

Example: $\displaystyle\int \frac{3}{\sqrt{9-4x^2}}\,dx = 3\int \frac{1}{\sqrt{9-4x^2}}\,dx$

$$= 3\int \frac{1}{\sqrt{9\left(1-\frac{4}{9}x^2\right)}}\,dx$$

Check:

$9\left(1-\frac{4}{9}x^2\right) = 9-4x^2$

$$= 3\int \frac{1}{\sqrt{9\left(1-\left(\frac{2x}{3}\right)^2\right)}}\,dx$$

$$= 3\int \frac{1}{3\sqrt{\left(1-\left(\frac{2x}{3}\right)^2\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{1-\left(\frac{2x}{3}\right)^2}}\,dx$$

$u = \frac{2x}{3} \Rightarrow \frac{3}{2}du = dx$

$$= \frac{3}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{3}{2}\arcsin(u) + c$$

$$= \frac{3}{2}\arcsin\left(\frac{2x}{3}\right) + c$$

Example: $\int_0^{1/2} \frac{\sin(\arctan(t))}{1+t^2}\,dt$

$u = \arctan(t)$

$\Rightarrow \frac{du}{dt} = \frac{1}{1+t^2} \Rightarrow (1+t^2)\,du = dt$

So $\int_0^{1/2} \frac{\sin(\arctan(t))}{1+t^2}\,dt$

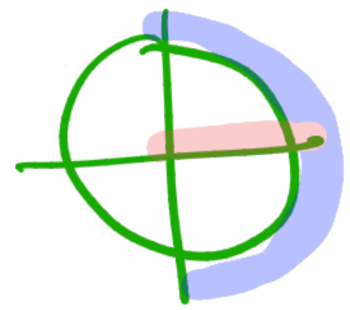
$= \int_{u=0}^{u=\arctan(1/2)} \frac{\sin(u)}{\cancel{1+t^2}} \cancel{(1+t^2)}\,du = \int_0^{\arctan(1/2)} \sin(u)\,du$

$= -\cos(u)\Big|_0^{\arctan(1/2)}$

$= -\cos(\arctan(1/2)) - (-\cos(0))$

$= -\frac{2}{\sqrt{5}} + 1$

- If $t=0$ then
  $u=\arctan(0) \Rightarrow$
  $\tan(u)=0$
  $\Rightarrow u=0$



- If $t = 1/2$ then
  $u=\arctan(1/2)$
  $(u \approx 0.46)$

$u=\arctan(1/2) \Rightarrow \tan(u) = 1/2$