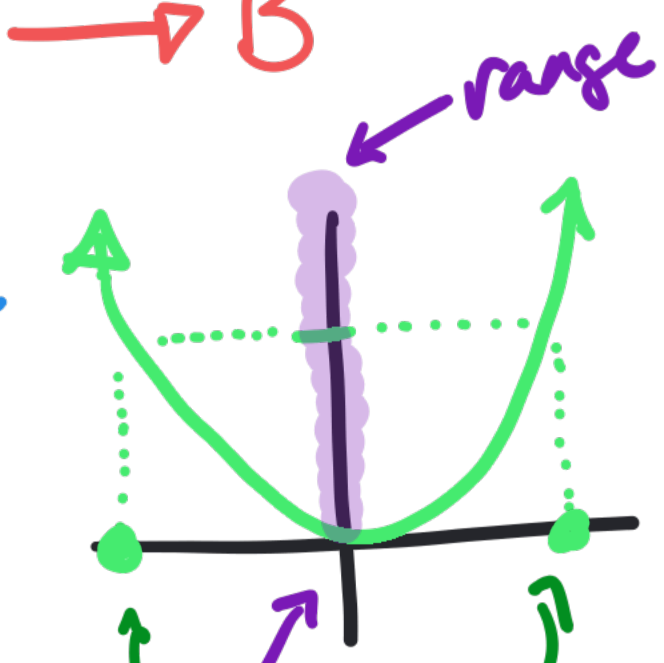
Intro Video: Section 1.5
Inverse Functions

Math F251X Calculus I

One-to-one and onto functions

$f : A \to B$



One-to-one: no element is hit twice

Onto: every element is hit at least once!

Nothing negative is hit: not onto!

different inputs, <u>same</u> output

NOT one-to-one

If $f: A \to B$ is one-to-one and onto, then there exists a function $g: B \to A$ such that

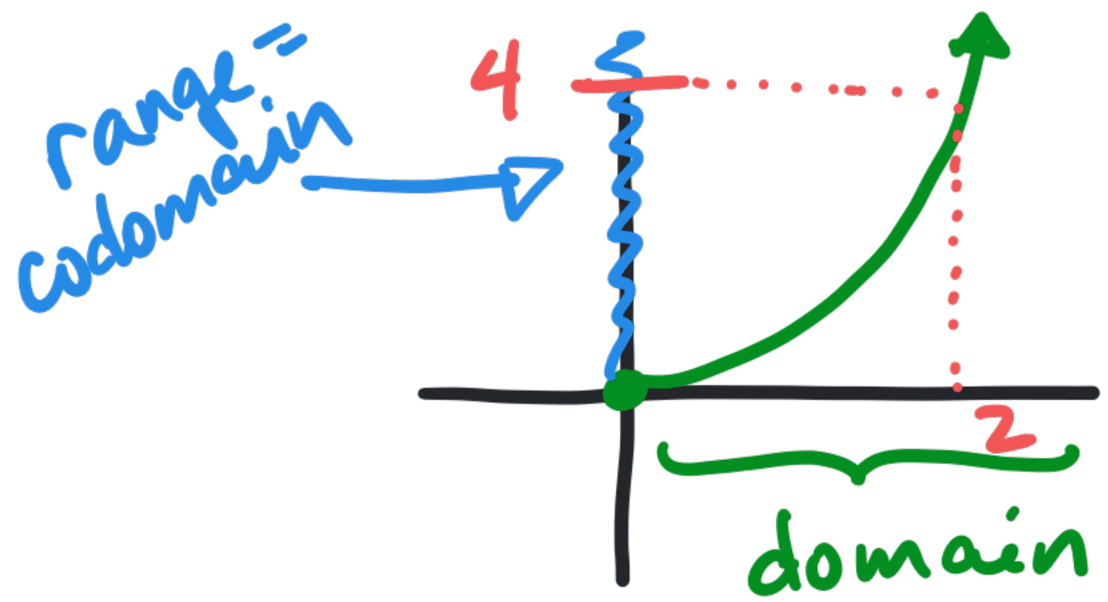
$$g(f(x)) = x$$

for <u>ALL</u> $x \in A$. We say $g = f^{-1}$ "f-inverse"

Note if $g(f(x)) = x$ for all $x \in A$, it is also true that $f(g(y)) = y$ for all $y \in B$.

Example:

$f : [0, \infty) \to [0, \infty)$ by $x \mapsto x^2$



What is $f^{-1}(4)$?

$\iff f(\,?\,) = 4$

$f(2) = 4$

$f^{-1}(x) = \sqrt{x}$ ← domain for this is $[0, \infty)$ = the range of the original function

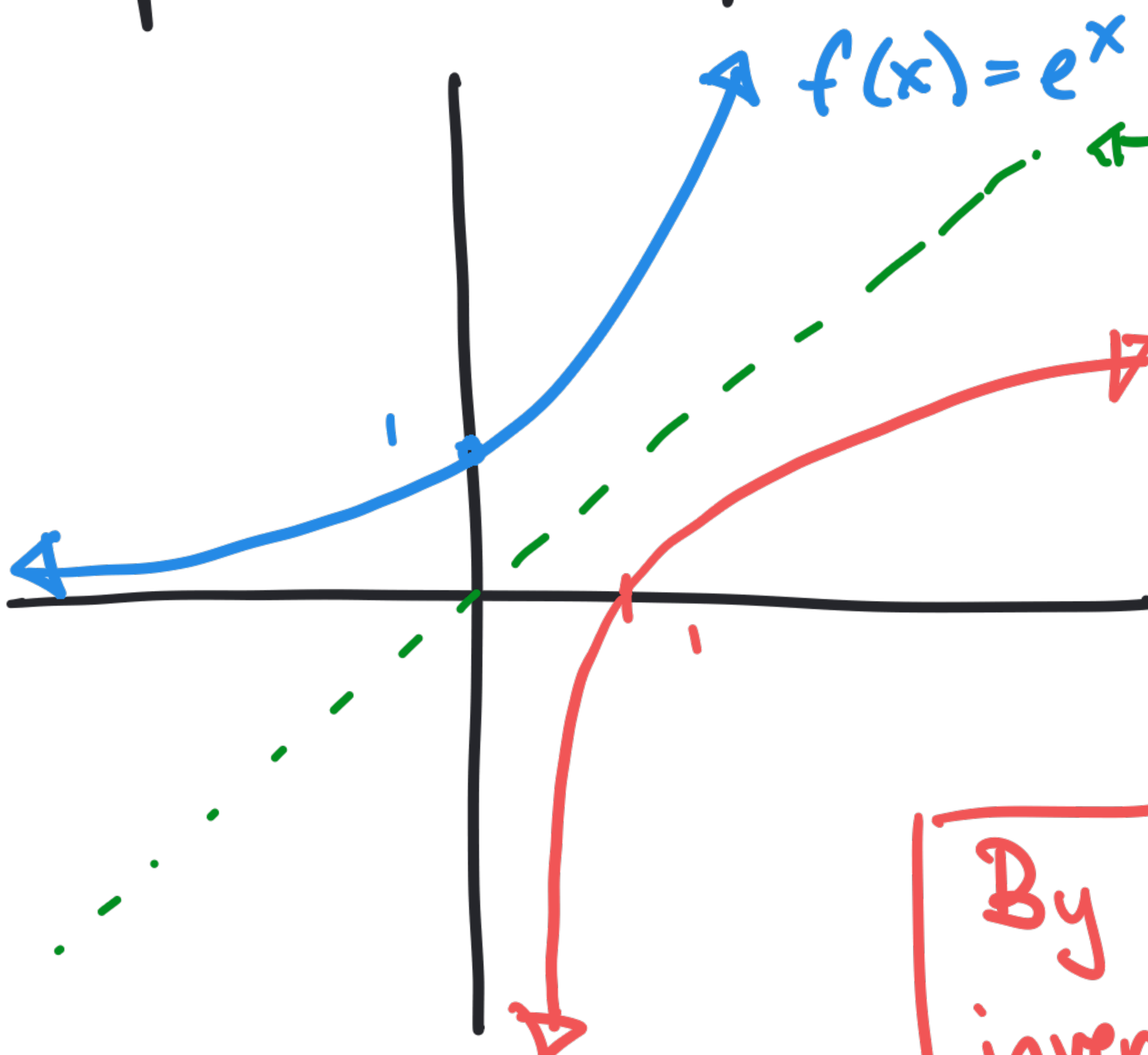
Example:

If $f(x) = x - 5$, what is $f^{-1}(20)$? $\boxed{f^{-1}(20) = 25}$

What input gives us an output of 20?

When $x = 25$, $f(25) = 20$

Important example: $f(x) = e^x$ and $g(x) = \ln(x)$



range of $f(x) = e^x$ is $(0, \infty)$

By definition, $\ln(x)$ is the inverse of e^x.

$e^{\ln(x)} = x$

$\ln(e^x) = x$

Domain of $\ln(x)$ is $(0, \infty)$.

Inverse of a^x is $\log_a(x)$

Example:

If $y = \log_{10}\left(\frac{1}{1000}\right)$, what is y?

$\iff 10^y = \frac{1}{1000} = 10^{-3}$, so $y = -3$.

Laws of logarithms:

1) $\log_b(xy) = \log_b(x) + \log_b(y)$

2) $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

3) $\log_b(x^r) = r\log_b(x)$.

Example: Write as a single logarithm: $\ln(b) + 2\ln(c) - 5\ln(d)$

$= \ln(b) + \ln(c^2) + \ln(d^{-5})$

$= \ln(b(c^2)(d^{-5})) = \ln\left(\frac{bc^2}{d^5}\right)$

Example: Solve $\ln(x) + \ln(x-1) = 1$ (for x).

$$\ln(x) + \ln(x-1) = 1 \implies$$

$$\ln\big((x)(x-1)\big) = 1 \implies$$

$$e^{\ln(x(x-1))} = e^1 \implies$$

$$x(x-1) = e \implies$$

$$x^2 - x - e = 0 \implies$$

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-e)}}{2}$$

$$= \frac{1 \pm \sqrt{1+4e}}{2}$$

$$x = \frac{1+\sqrt{1+4e}}{2} \quad \approx 2.223$$

or

$$\cancel{x = \frac{1-\sqrt{1+4e}}{2}} \quad \approx -1.223$$

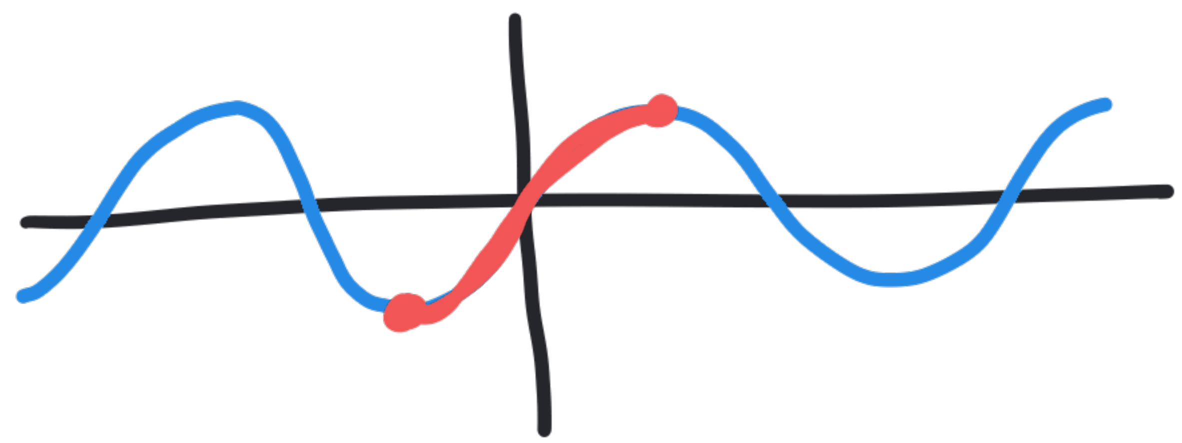
Solution:

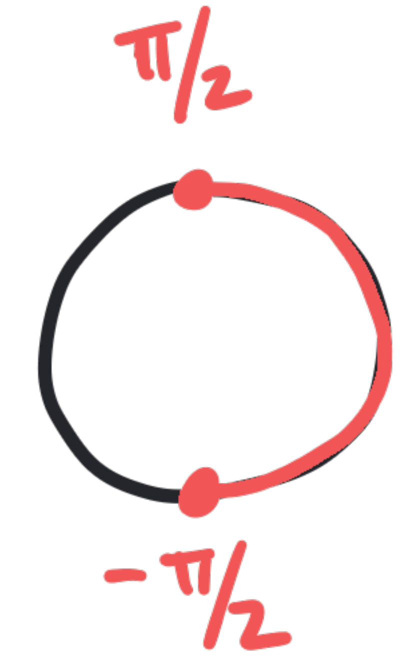
$$x = \frac{1+\sqrt{1+4e}}{2}$$

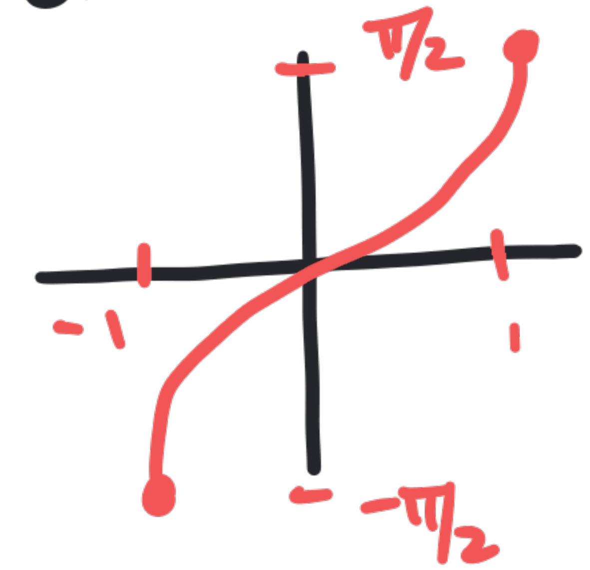
Inverse Trigonometric Functions:

$\sin^{-1}(x) = \arcsin(x)$

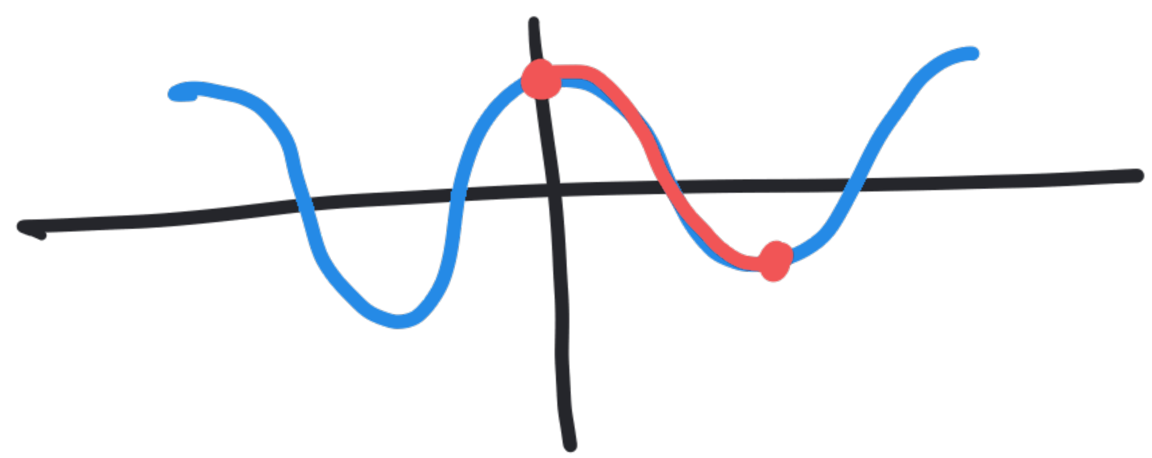
$\sin(x)$

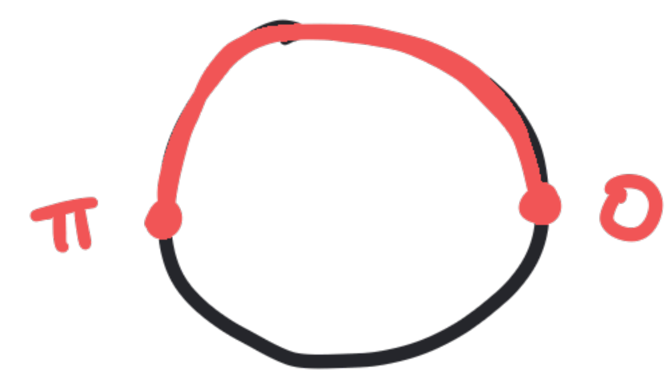




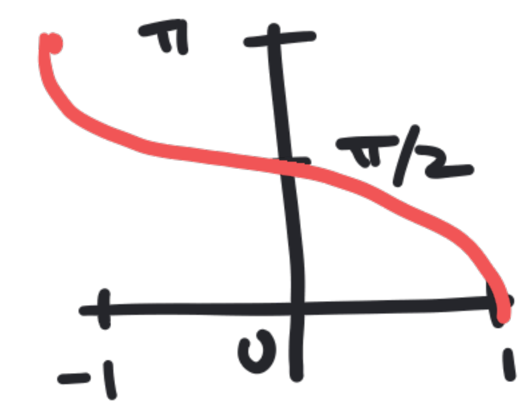


$\cos(x)$

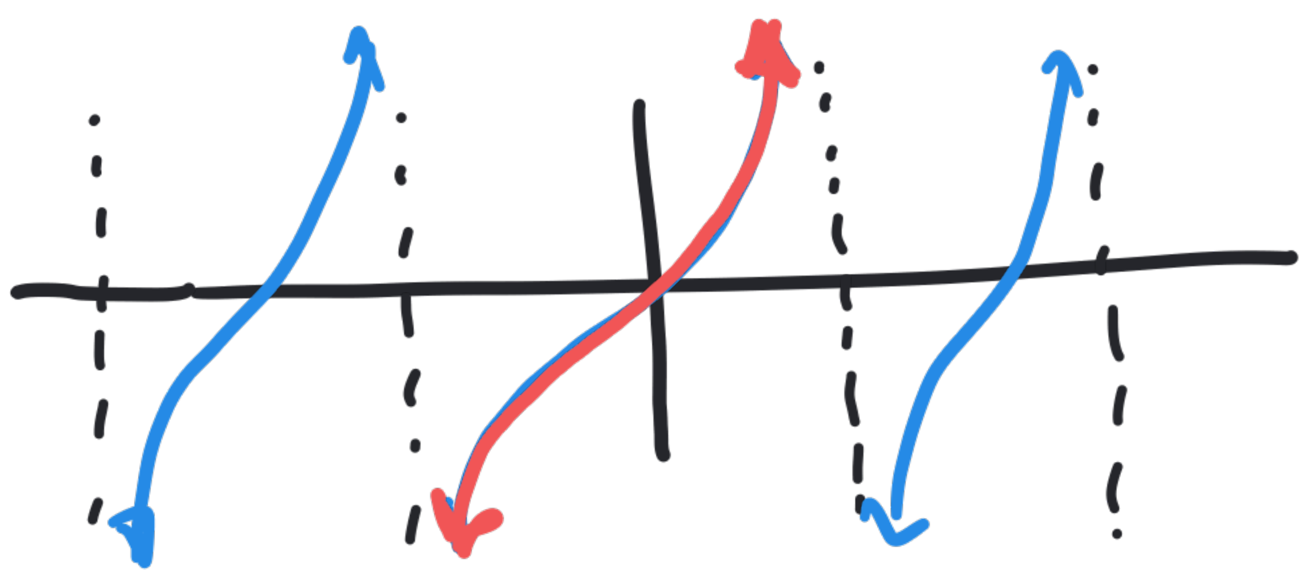


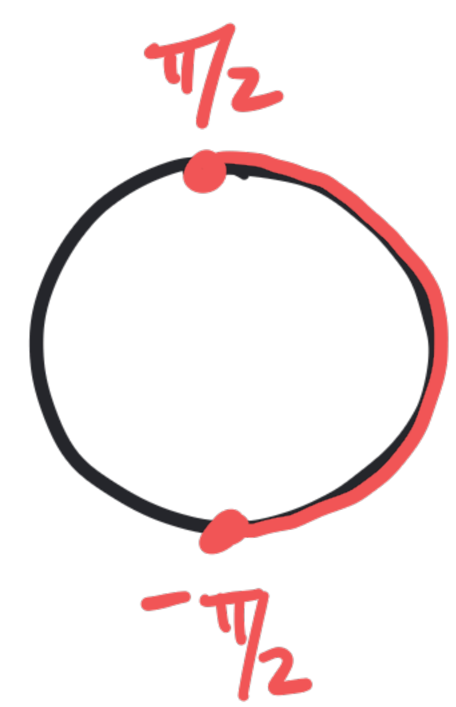


$\cos^{-1}(x) =$
$\arccos(x)$

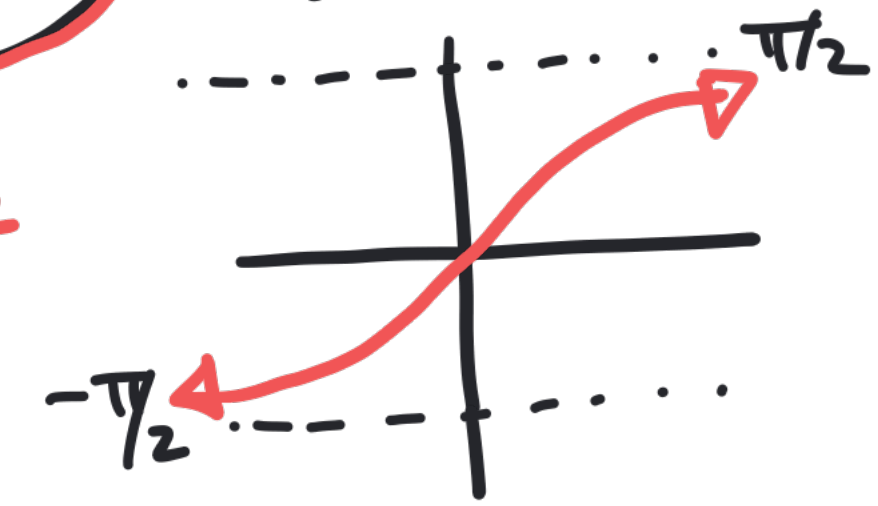


$\tan(x)$





$\tan^{-1}(x) =$
$\arctan(x)$



Warning: Bad notational collision!

$f^{-1}(x) \leftarrow$ inverse function

$\sin^2(x) \leftarrow$ shorthand for $(\sin(x))^2$

$a^{-1} = \frac{1}{a} \leftarrow$ usual exponent behavior

What does $\sin^{-1}(x)$ mean?

Your textbook: inverse to $\sin(x)$ and NOT $\frac{1}{\sin(x)}$

$$\frac{1}{\sin(x)} = \csc(x)$$

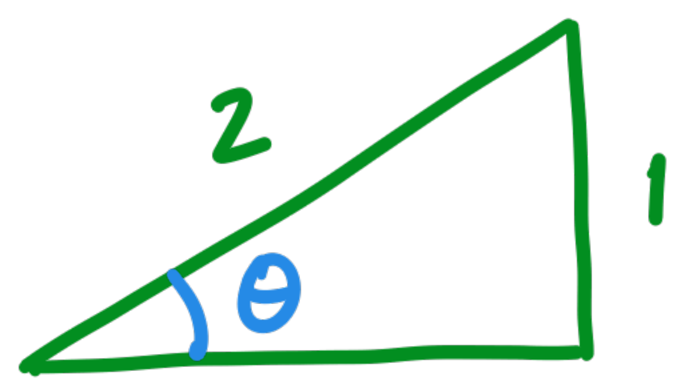
I will use $\boxed{\arcsin(x)}$ to mean the inverse function

(that is, $\arcsin(x) = \sin^{-1}(x)$.)

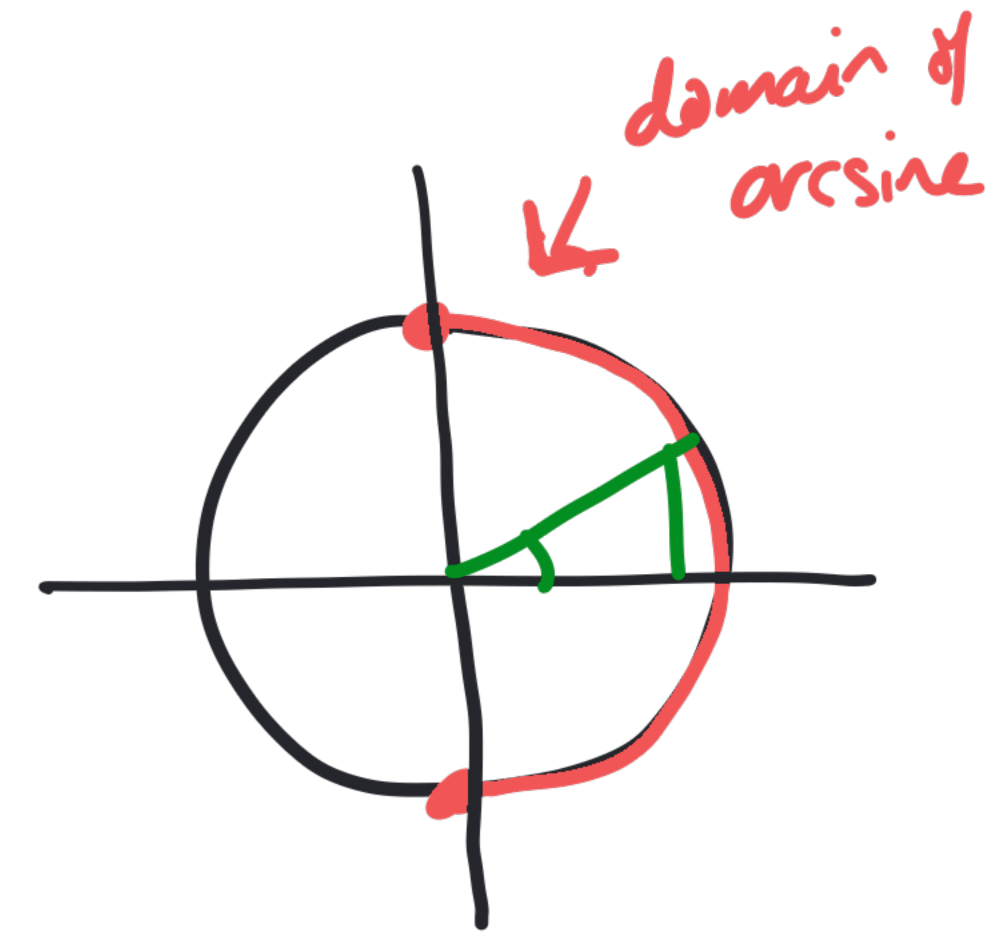
Examples:

① What is $\arcsin(1/2)$?

$\arcsin(1/2) = \theta \iff \sin(\theta) = \frac{1}{2}$



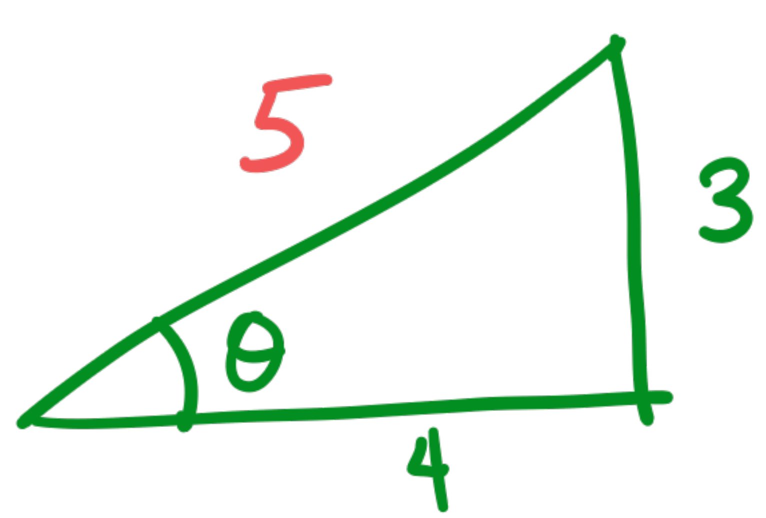
$\sin(\theta) = \frac{\text{opp}}{\text{hyp}}$



So $\theta = \pi/6$

② What is $\cos(\arctan(\frac{3}{4}))$?

$\theta = \arctan(3/4)$; what is $\cos(\theta)$?



← in this triangle, what is $\cos(\theta)$?

$\cos(\theta) = \frac{4}{5}$ So therefore,

$\cos(\arctan(\frac{3}{4})) = 4/5$.