# Final Review - Chapter 4
# (Applications of Differentiation)

- Find critical numbers of a function.
- Find the absolute maximum and absolute minimum of a function on a closed interval.

**Example 1:** Find the absolute maximum and minimum of $f(x) = xe^{x/2}$ on $[-3, 1]$

Find crit.#'s for f(x):

$f' = 1 \cdot e^{x/2} + x \cdot e^{x/2} \cdot \frac{1}{2}$

$= e^{x/2}\left(1 + \frac{1}{2}x\right) = 0$

$x = -2$. (Recall $e^{x/2} \neq 0$.)

Check: $-2$ does lie in $[-3,1]$

Chart:

| $x$ | $-3$ | $-2$ | $1$ |
|---|---|---|---|
| $f(x)$ | $\frac{-3}{e^{3/2}}$ | $\frac{-2}{e}$ | $e^{1/2}$ |

definitely larger than $-1$.

close to $-1$

so min.

positive. so max

* Note $f'(x)$ confirms this conclusion, since $f' < 0$ for $x < -2$ and $f' > 0$ on $x > -2$.

answer: maximum of f on $[-3,1]$: $e^{1/2}$

minimum of f on $[-3,1]$: $-2/e$

**Example 2:** Find the absolute maximim and minimum of $f(x) = \frac{x}{3} - \sqrt[3]{x}$ on the interval $[-1, 8]$

Find crit #'s for $f(x) = \frac{1}{3}x - x^{1/3}$

$f'(x) = \frac{1}{3} - \frac{1}{3}x^{-2/3} = \frac{1}{3}\left(1 - x^{-2/3}\right)$

$f' = 0$ when $1 - x^{-2/3} = 0$ or

$x^{2/3} = 1$. So $x = \pm 1$.

$f'$ is undefined when $x = 0$.

crit #s: $-1, 0, 1$.

Check: $-1, 0, 1$ all fall in interval

Chart:

| $x$ | $-1$ | $0$ | $1$ | $8$ |
|---|---|---|---|---|
| $f(x)$ | $\frac{2}{3}$ | $0$ | $-\frac{2}{3}$ | $\frac{2}{3}$ |

$f(-1) = -\frac{1}{3} - \sqrt[3]{-1}$

$= -\frac{1}{3} + 1 = \frac{2}{3}$

$f(8) = \frac{8}{3} - 2 = \frac{8-6}{3} = \frac{2}{3}$

answer: maximum of $f(x)$ on $[-1,8]$ is $2/3$.

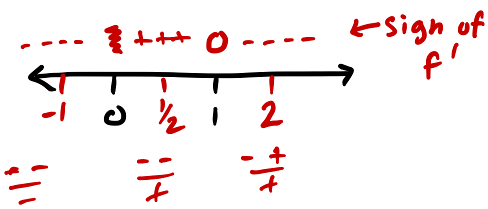
minimum of $f(x)$ on $[-1,8]$ is $-2/3$.

- Determine where a function is increasing decreasing.
- Determine where a function is concave up and concave down.

**Example 3:** Given $G(x) = 5x^{2/3} - 2x^{5/3}$ ; $G' = -\left(\frac{10(x-1)}{3x^{1/3}}\right)$; $G'' = \frac{-10(2x+1)}{9x^{4/3}}$

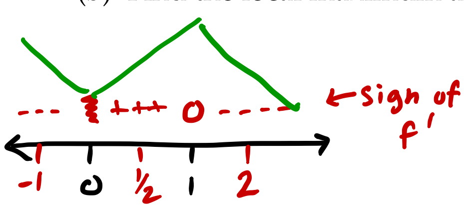
(a) Find the intervals of increase/ decrease.

crit #'s : $f' = 0$ when $x = 1$;
$f'$ undefined when $x = 0$;




G is increasing on $(0, 1)$ and decreasing on $(-\infty, 0) \cup (1, \infty)$.

(b) Find the local maximum and minimum values.



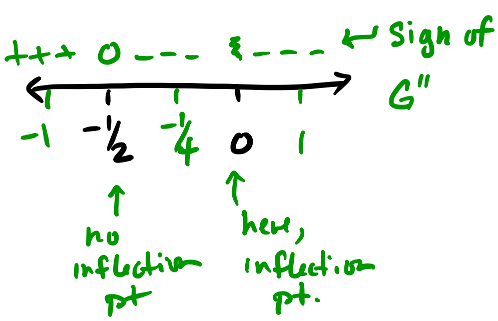

answer: $G(0) = 0$ is a local minimum
$G(1) = 5 - 2 = 3$ is a local maximum

(c) Find the intervals of concavity and the inflection points.

using $G'' = \frac{-10(2x+1)}{9x^{4/3}}$ we see:
← denominator always +'v !

$G'' = 0$ when $x = -1/2$

$G''$ undefined when $x = 0$.




answer :

G is concave up on $(-\infty, -\frac{1}{2})$ and concave down on $(-\frac{1}{2}, \infty)$.

$\left(-\frac{1}{2}, G(-\frac{1}{2})\right)$ is an inflection point.

$-(x^2+9)$ ← always −

doesn't factor

$2x(x^2+27)$

always +.

**Example 4:** Given $g(x) = \dfrac{x}{x^2-9}$, $g'(x) = \dfrac{-x^2-9}{(x^2-9)^2}$ and $g''(x) = \dfrac{2x^3+54x}{(x^2-9)^3}$ find the following.

always +

(a) Determine the intervals of increase/ decrease.

$g'$ is never 0.

$g'$ undefined at $x = \pm 3$

$g'(x) < 0$ for all $x$'s in its domain...

answer: $g(x)$ is decreasing on $(-\infty,3) \cup (-3,3) \cup (3,\infty)$ and increasing never.

(b) Find the relative maxima/ minima and indicate whether it is a maxima or minima. If there are no local exrema explain why.

$g(x)$ has no local extrema because it is always decreasing.

(c) Determine the intervals of concavity.

$g'' = 0$ when $x = 0$

$g''$ is undefined when $x = \pm 3$

--- ⁞ +++ 0 --- ⁞ +++ ← sign of $g''$

-4 -3 -1 0 1 3 4

answer: $g$ is ccave up on $(-3,0) \cup (3,\infty)$

ccave down on $(-\infty,-3) \cup (0,3)$.

(d) Find the inflection points. If there are no inflection points, explain why.

$(0,0)$ is the only inflection point.

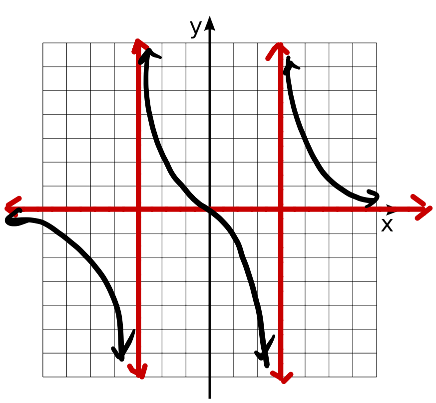
(e) State the vertical asymptotes for $g(x)$

$x = 3$, $x = -3$

(f) State the horizontal asymptotes for $g(x)$

$y = 0$

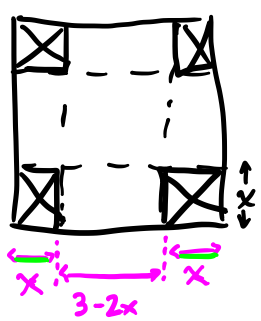
(g) Sketch the graph of $g(x)$.

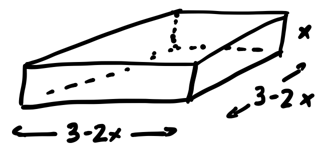


- Solve max/ min optimization problems.

**Example 5:** A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

maximize volume!



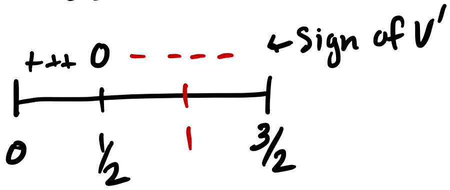

$V = (3-2x)^2 x = (9-12x+4x^2)x$

$= 9x - 12x^2 + 4x^3$

$V'(x) = 9 - 24x + 12x^2 = 3(3-8x+4x^2)$

$= 3(3-2x)(1-2x) = 0$

$x = \frac{3}{2}$ or $\frac{1}{2}$

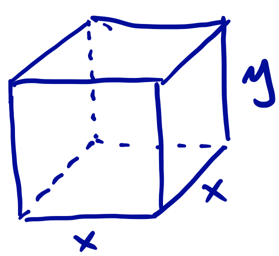



↑ max at $x = \frac{1}{2}$

$V(\frac{1}{2}) = (3 - 2\cdot\frac{1}{2})^2 \cdot \frac{1}{2} = 2^2 \cdot \frac{1}{2} = 2 \text{ ft}^3$

$$\begin{array}{r} 32 \\ \times 4 \\ \hline 128 \end{array}$$

**Example 6:** Suppose a box with a square base and open top must have a volume of 32 m$^3$. Find the dimensions of the box that minimize the amount of material used.




$V = x^2 y = 32$

So $y = 32x^{-2}$

$S = x^2 + 4xy$

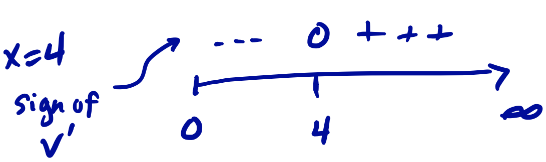
$S(x) = x^2 + 4x \cdot 32x^{-2}$

$= x^2 + 128x^{-1}$

$S'(x) = 2x - 128x^{-2} = 0$

$2x = \frac{128}{x^2}$ or $x^3 = \frac{128}{2} = 64 = 4^3$

So $x = 4$




So S has a minimum @ $x = 4$

answer:

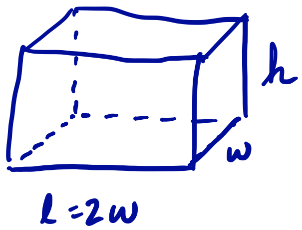
dimensions:

$x$ = base = 4 m

$y$ = height = 2 m

**Example 7:** A rectangular storage container with an open top is to have a volume of 10 m$^3$. The length of the base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$ 6 per square meter. Find the costs of materials for the cheapest such container.

$V = 10 \text{ m}^3$




base \$10/m$^2$

sides \$6/m$^2$

minimize cost

$V = 10 = lwh$
$= 2w^2h$

So $h = 5w^{-2}$

$C = 10 \cdot lw + 6 \cdot (2lh) + 6 \cdot (2wh)$ $\quad l = 2w$, $h = 5w^{-2}$

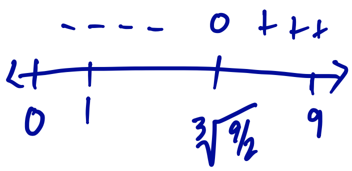
$= 10(2w)w + 12(2w)(5w^{-2}) + 12w \cdot 5w^{-2}$

$C(w) = 20w^2 + 120w^{-1} + 60w^{-1} = 20w^2 + 180w^{-1}$

$C'(w) = 40w - 180w^{-2} = 0$

$40w = \frac{180}{w^2}$ or $w^3 = \frac{180}{40} = \frac{9}{2}$

$w = \sqrt[3]{9/2}$ is the location of a minimum.




min Cost: $C(\sqrt[3]{9/2})$

$= 20\left(\frac{9}{2}\right)^{2/3} + 180 / \sqrt[3]{9/2}$

**Suggested Probs:** Ch. 4 Rvw. Pg 359 # 1, 5, 7, 9, 15, 17, 19, 27 & Pg 337 # 15, 19