RECITATION: 1-2 MATHEMATICAL MODELS: A CATALOG OF ESSENTIAL FUNCTIONS

Linear Models

If we have two variables that are "linearly related" or "y is a linear function of x" we can assume that the graph of the function is a line. This means that we can use all the stuff we know about lines to come up with an formula that "models" the situation.

Example 1: Recent studies indicate that the average surface temperature of earth has been rising steadily. Some scientists modeled the temperature by the linear function $T = \underbrace{0.02}_{m}t + \underbrace{8.50}_{b}$, where T is temperature in C° and t represents years since 1900.

(a) What do the slope and T-intercept represent?

$m = 0.02$ $\Delta T/\Delta t$ ← as time (t) increases by 1 year, temperature goes up $0.02\,°C$.

$b = 8.50$ ← global avg temp in °C in 1900.

(b) Use the equation to predict the average global surface temperature in 2100.

$t = 200$ years past 1900

$$T(200) = 0.02(200) + 8.50$$
$$= 4 + 8.5$$
$$= \boxed{12.5\,°C}$$

Two Forms of Line

- Slope-intercept: $\boxed{y = mx + b}$
- Point-slope: $\boxed{y - y_1 = m(x - x_1)}$

Example 2: Find an equation of the line that has the following properties.

(a) x-intercept 7 and y-intercept -2.

$(7, 0)$ $(0, -2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 0}{0 - 7} = 2/7$$

$$y - (-2) = \frac{2}{7}(x - 0)$$

$$\boxed{y = \frac{2}{7}x - 2}$$

(b) parallel to $y = 5 - 2x$ and passes through $(3, -5)$

$m = -2$

$$y - y_1 = m(x - x_1)$$
$$y + 5 = -2(x - 3)$$
$$y + 5 = -2x + 6$$
$$\boxed{y = -2x + 1}$$

Example 3: The monthly cost of driving a car depends on the number of miles driven. In May it cost \$380 to drive 480 miles and in June it cost \$460 to drive 800 miles. Express the monthly cost C as a function of the distance driven d, assuming a linear relationship gives a reasonable model. What does the slope represent? What does the C-intercept represent? Let $x = d$, $y = C$

points: $(480, 380)$ and $(800, 460)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{460 - 380}{800 - 480} = \frac{80}{320} = \boxed{\frac{1}{4}}$$

← cost per mile. \$0.25 per mi

$$y - y_1 = m(x - x_1)$$

$$y - 380 = \frac{1}{4}(x - 480)$$

$$y = \frac{1}{4}x - 120 + 380$$

$$y = \frac{1}{4}x + 260$$

$$\boxed{C = \frac{1}{4}d + 260}$$

C-intercept is cost to drive 0 miles, \$260

Often you are given data in a table and asked to decide what kind of function best "fits" the data. There are many statistical methods used to "fit" lines to data. We don't ask you to perform any of these statistical calculations in this course. However, it is a good idea to know what certain functions are called and be able to identify what their graphs could look like.

Polynomial Functions

Example 3: A ball is dropped from the upper observation deck of the CN Tower 450 m above the ground. The height above the ground h after t seconds is given by the equation $h(t) = -4.9t^2 + 0.96t + 449.36$.

(a) When does the ball hit the ground?

when $h(t) = 0$

$$0 = -4.9t^2 + 0.96t + 449.36$$

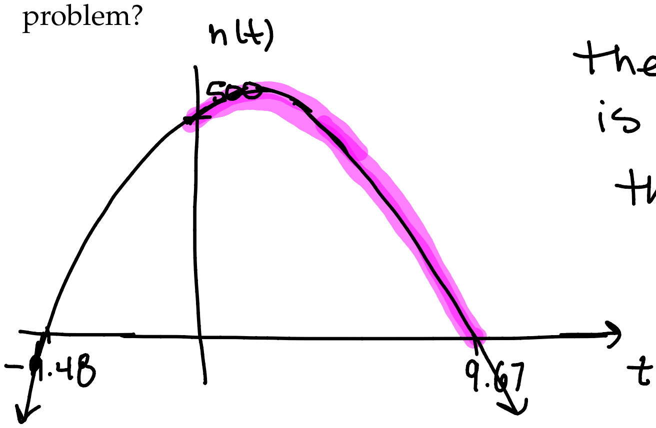
← does not factor, at least not easily, you need to use the quadratic formula

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-0.96 \pm \sqrt{0.96^2 - 4(-4.9)(449.36)}}{2(-4.9)}$$

$$\approx -9.48, \boxed{9.67 \text{ seconds}}$$

(b) Sketch a rough picture of the $h(t)$. What is the domain of $h(t)$ in the content of this problem?



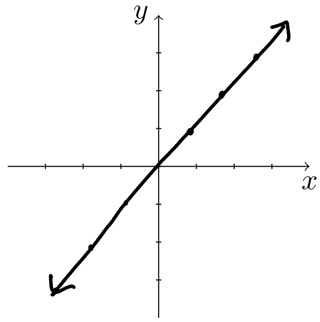
the domain of $h(t)$ is $0 \le t \le 9.67$, the part in pink.

General Graphs

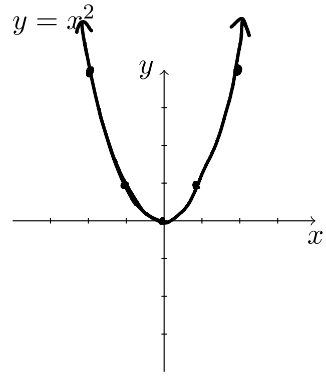
You should have the graphs of the following 12 functions down by heart. Graph the following functions. Clearly indicate any asymptotes, or interesting behavior.

Example 4: Sketch the graphs of the following functions:

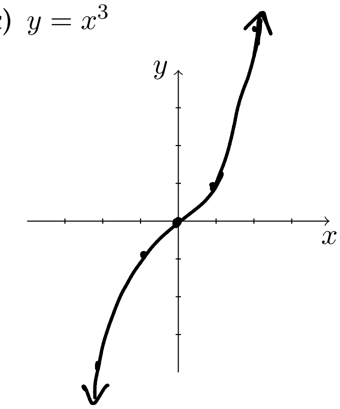
(a) $y = x$



(b) $y = x^2$

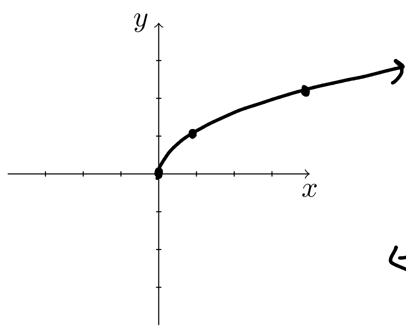


(c) $y = x^3$

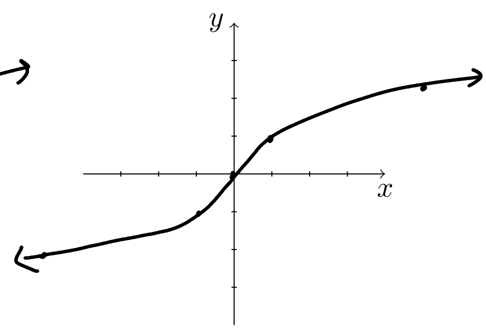


Example 5: Sketch the graphs of the following functions:

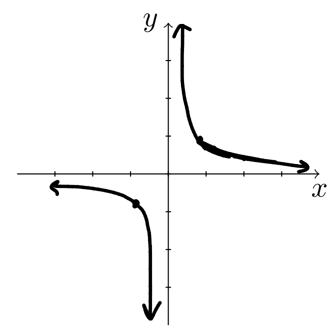
(a) $y = \sqrt{x}$



(b) $y = \sqrt[3]{x}$

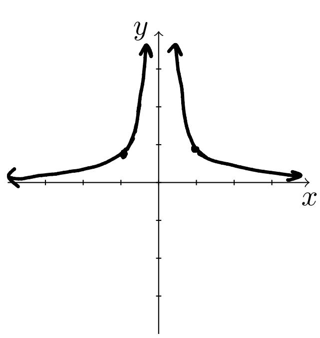


(c) $y = \dfrac{1}{x}$

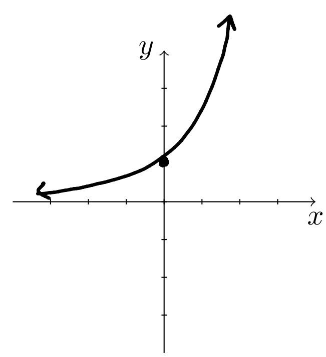


Example 6: Sketch the graphs of the following functions:

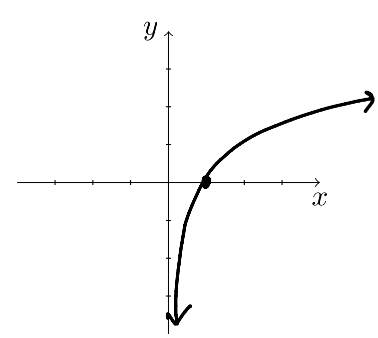
(a) $y = \dfrac{1}{x^2}$



(b) $y = e^x$

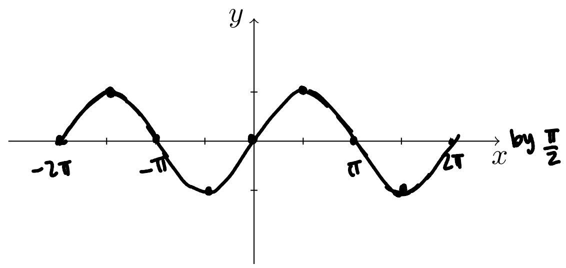


(c) $y = \ln x$

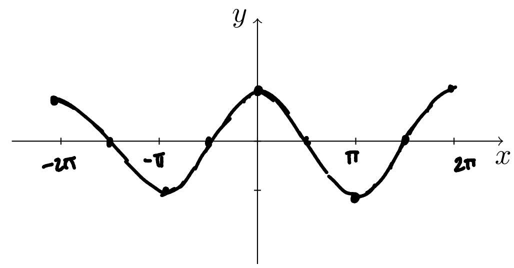


Example 7: Sketch the following functions on $[-2\pi, 2\pi]$

(a) $y = \sin x$



(b) $y = \cos x$



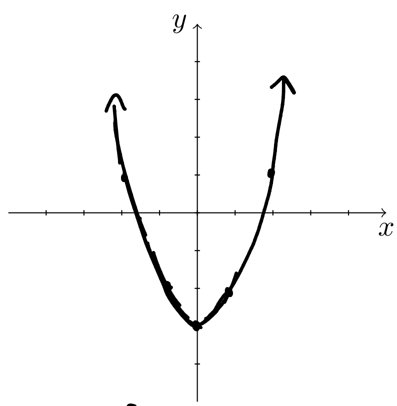
Using Transformations to Obtain Graphs of Related Functions

One can use the graphs of the functions in Examples 4 - 7 to obtain graphs of related functios if some basic principles of transformations are understood. Graph the following functions by plotting points. How do the graphs of these function relate to the parent function $f(x) = x^2$?

Example 8: Graph the following functions by plotting points.

a) $y = x^2 - 3$

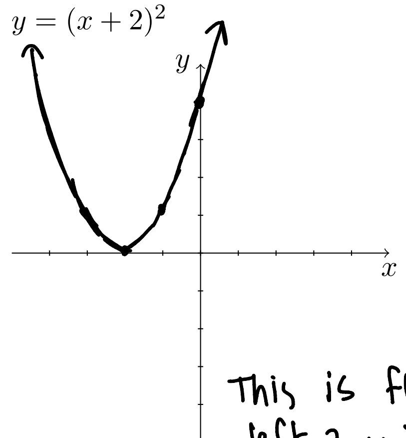
x	y
+2	1
+1	-2
0	-3
-1	-2
-2	1



note this is $f(x)=x^2$ shifted down two units.

b) $y = (x+2)^2$

x	y
−4	4
−3	1
−2	0
−1	1
0	4



This is $f(x)=x^2$ shifted left 2 units.

From this example, can you generalize to how the graph of a function $y = f(x)$ will relate to the following functions, given some positive constant $c > 0$?

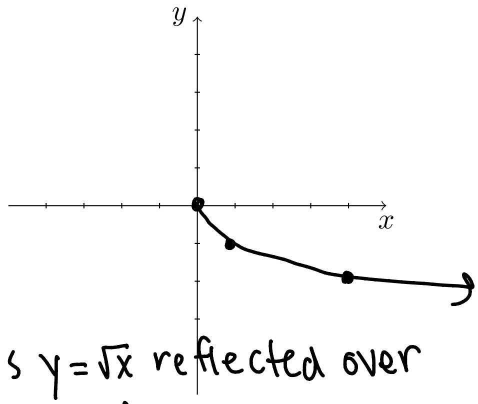
Vertical and Horizontal Shifts Let $c > 0$. To obtain the graph of

- $y = f(x) + c$, shift the graph of $y = f(x)$ up c ↑.
- $y = f(x) - c$, shift the graph of $y = f(x)$ down c ↓.
- $y = f(x + c)$, shift the graph of $y = f(x)$ left c ←.
- $y = f(x - c)$, shift the graph of $y = f(x)$ right c →.

Example 9: Sketch the following functions. How do these function relate to the parent function $y = f(x) = \sqrt{x}$?

x	$y = -\sqrt{x}$
0	0
1	$-\sqrt{1} = -1$
4	$-\sqrt{4} = -2$

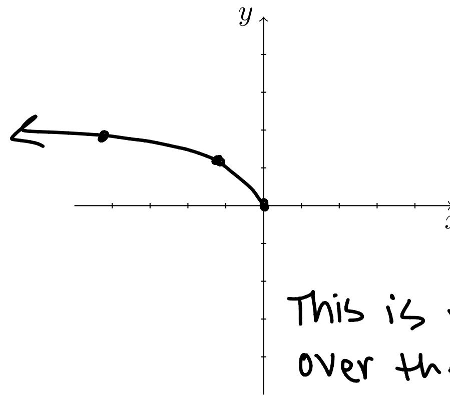
a) $y = -\sqrt{x}$



This is $y = \sqrt{x}$ reflected over the x-axis.

b) $y = \sqrt{-x}$

note you have to plug in negatives



x	$y = \sqrt{-x}$
0	0
-1	$\sqrt{-(-1)} = 1$
-4	$\sqrt{-(-4)} = 2$

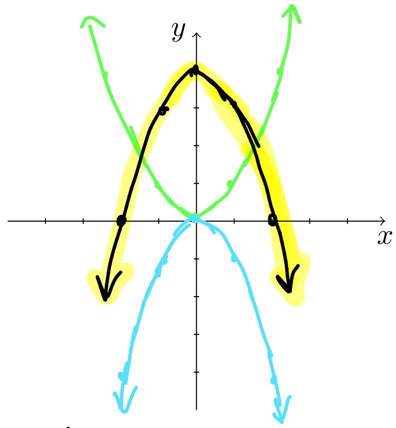
This is $y = \sqrt{-x}$ reflected over the y-axis.

Vertical and Horizontal Reflecting The following relationships can be generalized from Example 9.

- $y = -f(x)$, reflect the graph of $y = f(x)$ over the x-axis.
- $y = f(-x)$, reflect the graph of $y = f(x)$ over the y-axis.

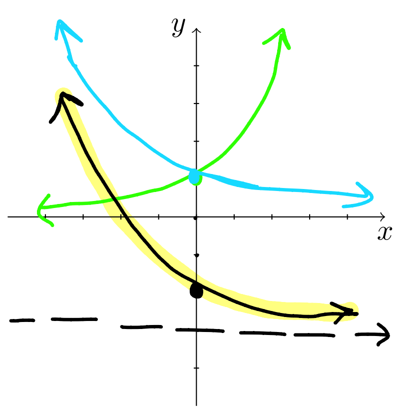
Example 10: Using the transformations you discovered (or re-learned) in Examples 7 and 9, Sketch the following functions using transformations. Begin by sketching in the parent function. Label any asymptotes. Then, describe the transformations that were required to turn the parent function into the function you have graphed.

(a) $f(x) = 4 - x^2 = -x^2 + 4$



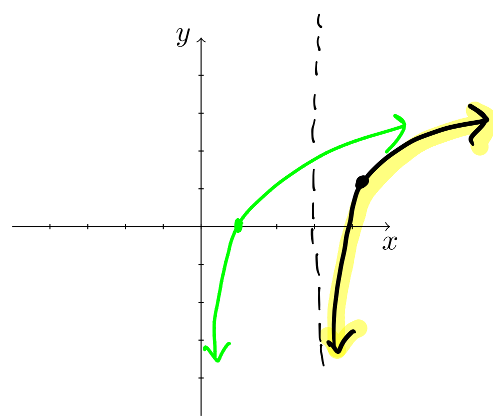
"parent" is $\boxed{y = x^2}$, reflected over the $\boxed{\text{x-axis}}$, shifted up 4 units

(b) $f(x) = e^{-x} - 3$



parent is $\boxed{y = e^x}$ reflect over $\boxed{\text{y-axis}}$; shifted down 3

(c) $f(x) = \ln(x - 3) + 1$



parent is $\boxed{y = \ln x}$ shifted right 3, up 1