Recitation 10: 4-2 How Derivatives Affect the Shape of a Graph (part 2)

Warm-up Questions:

1. Given a function $y = f(x)$ how do you...

   (a) determine where f is increasing/decreasing?

   - find f'.
   - In the domain of $f(x)$, where $f'>0$, f is increasing; where $f'<0$, f is decreasing.

   (b) use f' to identify any local maximum and minimum values?

   - find critical points of f.
   - Check the sign-change of f'.
   - If f' changes $+$ to $-$, $\cap$, a local max. If f' changes $-$ to $+$, $\cup$, a local min.

   (c) determine where f is concave up or concave / down?

   - find f''
   - In the domain of f, where $f''>0$, f is concave up; where $f''<0$, f is concave down.

   (d) find inflection points?

   Find x-values in domain of f where the sign of f'' changes. To find the point you need to find the y value, too.

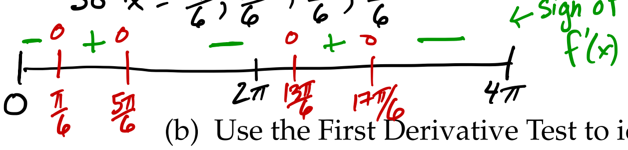
2. Let $f(x) = x - 2\cos x$ be restricted to the interval $I = [0, 4\pi]$.

   (a) Determine intervals of increase and decrease of f on I.

   $f'(x) = 1 + 2\sin x = 0$

   We need $\sin x = -\frac{1}{2}$.

   So $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$



   answer

   f increases on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right) \cup \left(\frac{13\pi}{6}, \frac{17\pi}{6}\right)$

   f decreases on $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \frac{13\pi}{6}\right) \cup \left(\frac{17\pi}{6}, 4\pi\right)$

   (b) Use the First Derivative Test to identify any local maximums or minimums of f on I.

   loc. max. at $x = 5\pi/6, \frac{17\pi}{6}$

   local maximum values: $\frac{5\pi}{6} + \sqrt{3}$ and $\frac{17\pi}{3} + \sqrt{3}$

   loc min at $x = \frac{\pi}{6}, \frac{13\pi}{6}, 0, 4\pi$

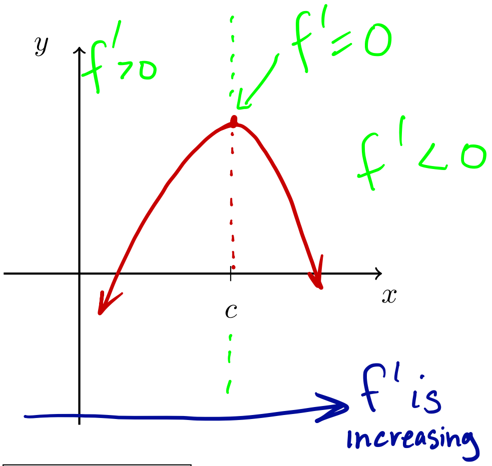
   local min. values: $\frac{\pi}{6} - \sqrt{3}$, $\frac{13\pi}{6} - \sqrt{3}$, -2, -2

   (c) Graph $f(x)$ on your calculator to check your answer is correct.

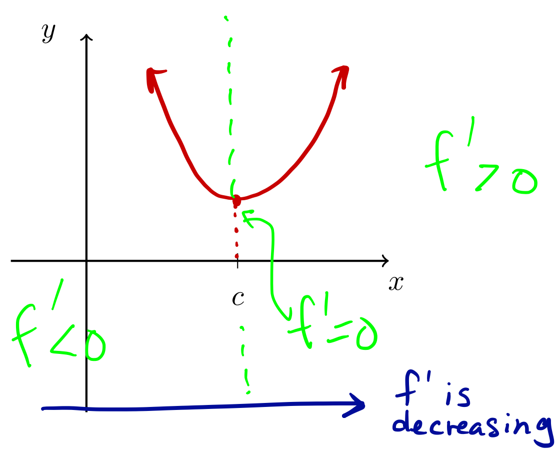
MOTIVATING EXAMPLES:

Assume $f(x)$ is differentiable (and therefore continuous) for all real numbers. On the axes below, sketch a graph of $f(x)$ with the given property.

(i) $f(x)$ has a local maximum at $x = c$



(ii) $f(x)$ has a local minimum at $x = c$



QUESTION 1: What can you say about $f'(c)$ in picture (i)? picture (ii)? Is it the same for your neighbors pictures? QUESTION 2: What can you say about $f''(c)$ in picture (i)? picture (ii)? Is it the same for your neighbors pictures?

f'' is positive in (i)

f'' is negative in (ii)

> THE SECOND DERIVATIVE TEST: Suppose f'' is continuous near c.
>
> a) If $f'(c)$ $\underline{= 0}$ and $f''(c)$ $\underline{> 0}$, then f has a local minimum at c. ∪ cc up
>
> b) If $f'(c)$ $\underline{= 0}$ and $f''(c)$ $\underline{< 0}$, then f has a local maximum at c. ∩ cc down

QUESTION 3: What happens if $f'(c) = 0$ and $f''(c) = 0$? Can you draw any conclusions about whether f has a local max or min? Why"

No.

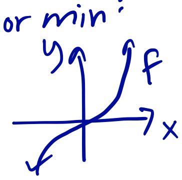
Example: $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

So $f'(0) = f''(0) = 0$.

f has no max or min:



Example: $g(x) = x^4$

$g'(x) = 4x^3$

$g''(x) = 12x^2$

So $g'(0) = g''(0) = 0$.

f has a min at $x=0$:

These two examples show that in the case $f' = f'' = 0$, anything could happen.

Example 2: Find the local maximum and minimum values of the functions. Choose either the first or second derivative test. Explain why you made the choice that you did.

(a) $f(x) = x^4 - 4x + 3$

$f'(x) = 4x^3 - 4 = 0$ when $x = 1$

$f''(x) = 12x$

$f''(1) = 12 \cdot 1 = 12 > 0$ ∪

So f has a local minimum of $f(1) = 1 - 4 + 3 = 0$ at $x = 1$ and no maximum.

Why choose 2nd der. test? f'' is simple to find.

(b) $f(x) = \dfrac{x}{x^2+2}$

$f'(x) = \dfrac{(x^2+2)(1) - x(2x)}{(x^2+2)^2} = \dfrac{2-x^2}{(x^2+2)^2} = 0$

when $x = \pm\sqrt{2}$

Sign of f': $-$ 0 $+$ 0 $-$ (at $-\sqrt{2}$, $\sqrt{2}$)

So $f(x)$ has a local minimum of $f(-\sqrt{2}) = -\sqrt{2}/2$ at $x = -\sqrt{2}$ and a local maximum of $\sqrt{2}/2$ at $x = \sqrt{2}$

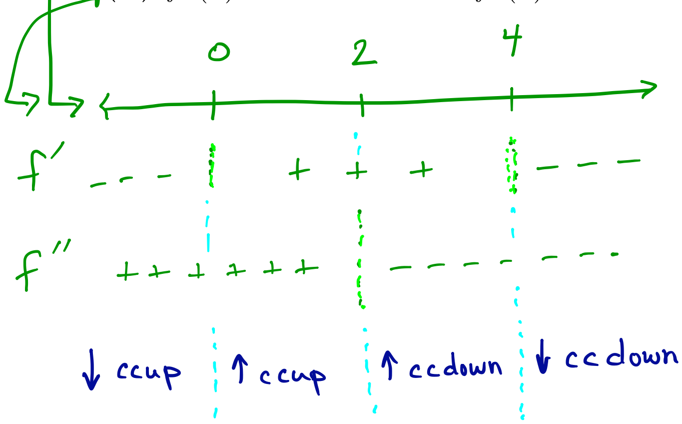
Why choose 1st der test?
Don't want to have to find f''. bleach!

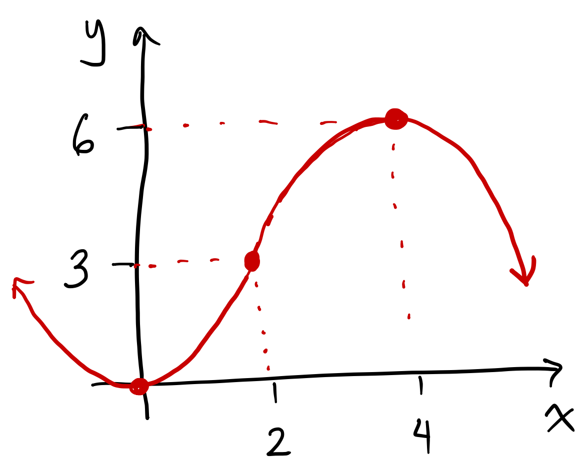
Example 3: Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f(0) = 0$, $f(2) = 3$, $f(4) = 6$, $f'(0) = f'(4) = 0$.

(ii) $f'(x) > 0$ for $0 < x < 4$ and $f'(x) < 0$ for $x < 0$ and for $x > 4$.

(iii) $f''(x) > 0$ for $x < 2$ and $f''(x) < 0$ for $x > 2$.





Example 4: Given the function $f(x) = \ln(x^2+4)$ find the following.

(a) Find the intervals of increase or decrease.

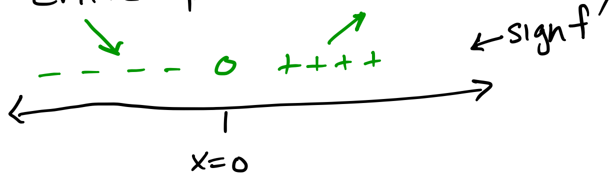
$$f'(x) = \frac{1}{x^2+4} \cdot 2x = \frac{2x}{x^2+4}$$

answer

f is increasing on $[0, \infty)$

f is decreasing on $(-\infty, 0)$

critical pts: $x=0$



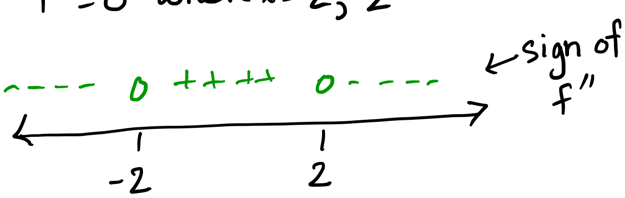
(b) Find the local maximum and minimum values.

local min at $x=0$. min value: $y = f(0) = \underline{\ln 4}$

(c) Find the intervals of concavity and inflection points.

$$f''(x) = \frac{(x^2+4)(2)-(2x)(2x)}{(x^2+4)^2} = \frac{2x^2+8-4x^2}{(x^2+4)^2} = \frac{8-2x^2}{(x^2+4)^2} = \frac{2(2-x)(2+x)}{(x^2+4)^2}$$

$f''=0$ when $x = 2, -2$

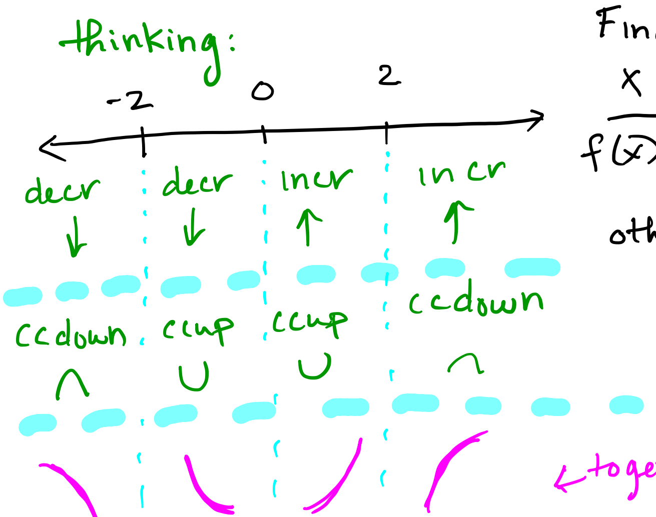


answer:

f is concave up on $(-2, 2)$ and concave down on $(-\infty, -2) \cup (2, \infty)$.

(d) Use the information to sketch the graph.

thinking:



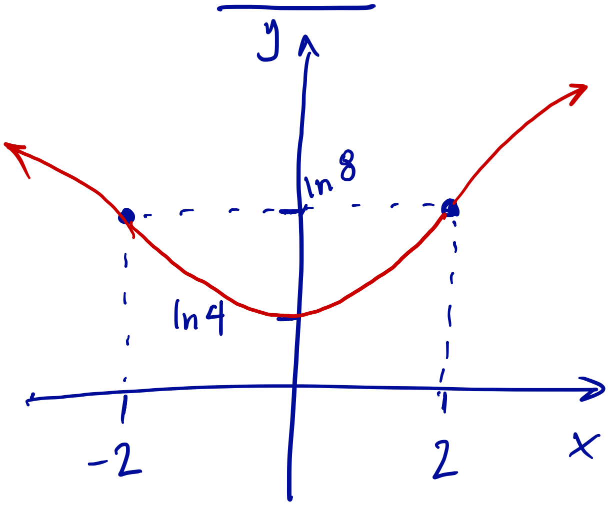
Find crucial points

x	-2	0	2
$f(x)$	$\ln 8$	$\ln 4$	$\ln 8$

other observations:

f is even. So symmetric across y-axis

answer



Example 5: Given the function $f(x) = 5x^{2/3} - 2x^{5/3}$, find the following.

(a) Find the intervals of increase or decrease.

$$f'(x) = \frac{10}{3}x^{-1/3} - \frac{10}{3}x^{2/3} = \frac{10}{3}\left(\frac{1-x}{x^{1/3}}\right)$$

ans:

f is increasing on $(0,1)$

and decreasing on $(-\infty,0) \cup (1,\infty)$

critical pts: $x=0$, $x=1$

Interval	$(-\infty,0)$	$(0,1)$	$(1,\infty)$
Sign of f'	$-$	$+$	$-$
incr/decr?	↓	↑	↓

Just FYI. Bad choice here. Easier to take derivative of $f' = \frac{10}{3}\left(x^{-\frac{1}{3}} - x^{\frac{2}{3}}\right)$.

(b) Find the local maximum and minimum values.

local minimum at $x=0$. minimum value is $f(0)=0$

local maximum at $x=1$. maximum value is $f(1)=3$

(c) Find the intervals of concavity and inflection points.

$$f''(x) = \frac{10}{3}\cdot\left(\frac{x^{1/3}(-1)-(1-x)\cdot\frac{1}{3}x^{-2/3}}{x^{2/3}}\right)\cdot\frac{x^{2/3}}{x^{2/3}} = \frac{10}{3}\left(\frac{-x-(1-x)\frac{1}{3}}{x^{4/3}}\right) = \frac{10}{9}\left(\frac{-3x-1+x}{x^{4/3}}\right)$$

$$= \frac{-10}{9}\left(\frac{1+2x}{x^{4/3}}\right).$$

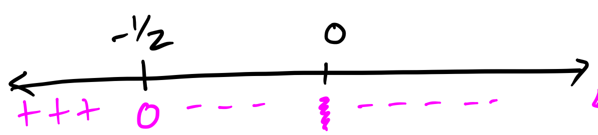
ANSWER:

f is concave up on $(-\infty, -\frac{1}{2})$

and concave down on $(-\frac{1}{2}, \infty)$

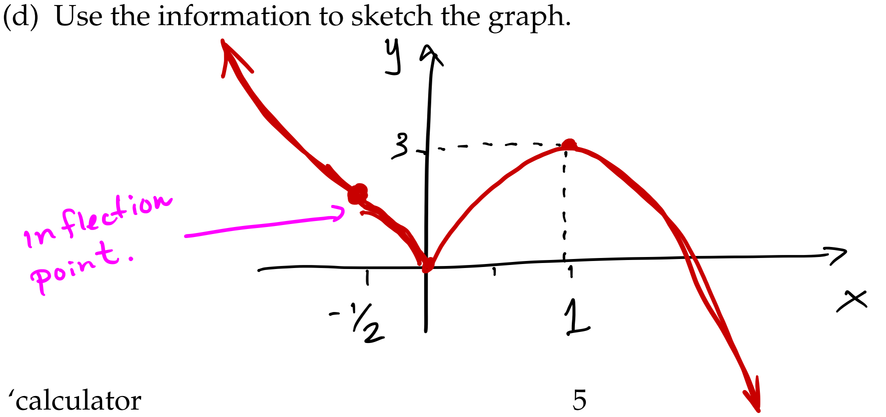
$f''=0$ when $x=-\frac{1}{2}$

f'' undefined when $x=0$



Sign f''

(d) Use the information to sketch the graph.



Example 6: Suppose the function $f(t) = t^3 - 12t + 2$ describes the motion of a particle along the t-axis for $t \geq 0$. Find $f'(2)$ and $f''(2)$. Is the velocity of the particle increasing or decreasing at $t = 2$?

Explain your answer in complete sentences.

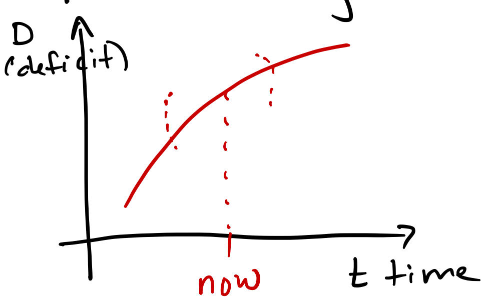
$f'(t) = 3t^2 - 12 \qquad f'(2) = 0$

$f''(t) = 6t \qquad f''(2) = 12 > 0$

Answer: The particle is speeding up. Since f'' is positive at $t=2$, we know velocity (f') is increasing. Increasing velocity means increasing speed, or, speeding up.

Example 7: An economist announces that the national deficit is increasing, but at a decreasing rate. Interpret this statement in terms of a function and its first and second derivatives.

picture in my head:

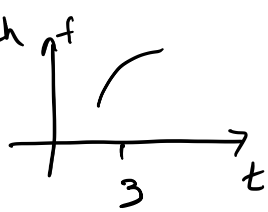


answer: If $D(t)$ gives the national debt over time (or, as a function of time), the economist says $D(t)$ is increasing. So $D'(t)$ is positive. But he/she also says D is increasing by less. So $D'(t)$ is decreasing. So $D''(t)$ is negative.

Example 8: Let $f(t)$ be the temperature at time t where you live and suppose at time $t = 3$ you feel uncomfortably cold. How do you feel about the given data in each case?

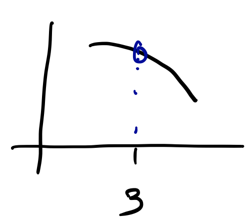
a) $f'(3) = 2$, $f''(3) = -4$

Sketch



ANS: On the up-side, $f' > 0$ means the temperature is getting warmer. On the down-side, it is getting warmer at a slower rate.

b) $f'(3) = -2$, $f''(3) = -4$



ANS: This is nothing but bad. The temperature is dropping ($f' < 0$) and it is dropping ever more quickly ($f'' < 0$).