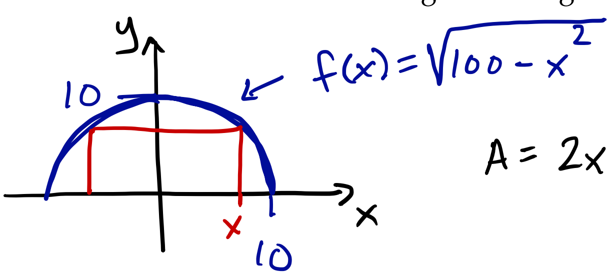
Recitation 11: 4-7 Optimization (Part 2)

Starter Problem: Write out the steps (hoops) you will need get through in order to produce a complete and correct answer to an optimization problem.

- Determine the $\boxed{\text{quantity}}$ to be maximized or minimized
- Write that $\boxed{\text{quantity}}$ as a function of one variable and determine its domain
- Find critical pts. use some test (closed-interval or 1st der or 2nd der) to identify where the max or min occurs.
- Answer the question

Practice Problems:

1. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 10.



We want to maximize area

$A = 2x \cdot y = 2x\sqrt{100-x^2}$ where $[0,10]$ is the domain

Closed-Interval Test

$A'(x) = 2 \cdot 1 \cdot (100-x^2)^{1/2} + \cancel{2}x \cdot \frac{1}{\cancel{2}}(100-x^2)^{-1/2}(-2x)$

$= 2\sqrt{100-x^2} - \dfrac{2x^2}{\sqrt{100-x^2}} = \dfrac{200-2x^2-2x^2}{\sqrt{100-x^2}}$

$= \dfrac{4(50-x^2)}{\sqrt{100-x^2}}$

Critical pt: $x = \sqrt{50} = 5\sqrt{2}$,

$x = 10$

x	0	10	$5\sqrt{2}$
A	0	0	$2 \cdot 5\sqrt{2} \cdot \sqrt{50} = 200$

Ans: The largest possible rectangle has area 200 units squared.

2. A retailer has been selling 1200 tablet computers a week for $350 each. The marketing department estimates that an additional 80 tablets will sell each week for every $ 10 that the price is lowered.

   (a) Find the demand function.

p = price

q = # tablets sold

pts : (1200, 350)

(1280, 340)

$$m = \frac{\Delta p}{\Delta q} = \frac{340-350}{1280-1200} = \frac{-10}{80} = -\frac{1}{8}$$

$$p - 350 = -\frac{1}{8}(q - 1200)$$

$$\boxed{p = -\frac{1}{8}q + 500}$$

   (b) What should the price be set at in order to maximize revenue?

$$R(q) = p \cdot q = \left(-\frac{1}{8}q + 500\right)q = -\frac{1}{8}q^2 + 500q \text{ , domain : } (0, \infty)$$

$$R'(q) = -\frac{1}{4}q + 500 = 0$$

$q = 2000$

$R'' = -\frac{1}{4} < 0$. R is cc down.

So a max occurs at $q = 2000$

Answer:

The price that will maximize revenue is

$$p = -\frac{1}{8}(2000) + 500 = \$250.$$

   (c) If the retailer's weekly cost function is $C(x) = 35,000 + 120x$ where x = # tablets what price should it choose in order to maximize its profit.

profit = revenue - cost

$$P(x) = -\frac{1}{8}x^2 + 500x - [35000 + 120x]$$

$$= -\frac{1}{8}x^2 + 380x - 3500$$

$$P'(x) = -\frac{1}{4}x + 380 = 0$$

$x = 4 \cdot 380 = 1520$

$P'' = -\frac{1}{4} < 0$. So P is cc down.

So P has a maximum at

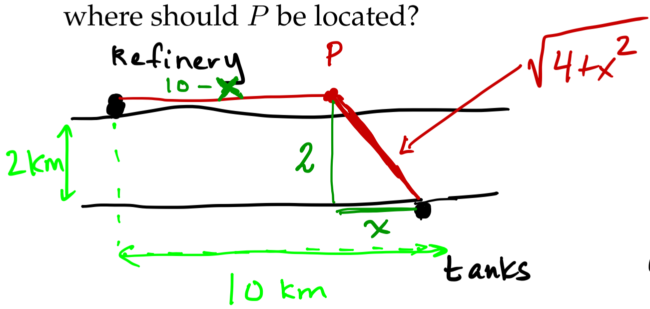
$x = 1520$.

To find price, plug into demand :

$$p = -\frac{1}{8}(1520) + 500 = \$310.$$

So charging \$310/tablet will maximize profit.

3. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 10 km downstream of the refinery. The cost of laying pipe is \$ 10,000/ km over land to a point P on the opposite bank and then \$ 40,000/km under the river to the tanks. To minimize the cost of pipeline, where should P be located?



Cost (in 10,000's)

$C(x) = 1\cdot(10-x) + 4\cdot(4+x^2)^{1/2}$

where x can take values $[0, 10]$.

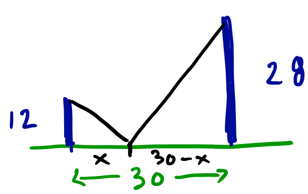
$C'(x) = -1 + 4\cdot\frac{1}{2}(4+x^2)^{-1/2}(2x) = \frac{4x}{\sqrt{4+x^2}} - 1$

Critical points: $4x = (4+x^2)^{1/2}$ or $16x^2 = 4+x^2$. Thus, $x^2 = 4/15$. So $x = 2/\sqrt{15}$.

x	0	10	$2/\sqrt{15}$
C	$10+8 = 18$	$4\sqrt{104} = 4\sqrt{26}$	17.45 ← min

Ans:
P should be located $10 - \frac{2}{\sqrt{15}}$ km down river from the refinery.

4. Two posts, on 12 feet high and the other 28 feet high stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?



$L(x) = (12^2 + x^2)^{1/2} + ((30-x)^2 + 28^2)^{1/2}$; domain: $[0, 30]$

$L'(x) = \cancel{\tfrac{1}{2}}(144+x^2)^{-1/2}(\cancel{2}x) + \cancel{\tfrac{1}{2}}((30-x)^2+28^2)^{-1/2}(\cancel{2}(30-x)(-1))$

$= \frac{x}{\sqrt{144+x^2}} - \frac{30-x}{\sqrt{(30-x)^2+28^2}}$

Critical pts:

$x((30-x)^2+28^2)^{1/2} = (30-x)(144+x^2)^{1/2}$

$x^2(900 - 60x + x^2 + 784) = (900 - 60x + x^2)(144+x^2)$

$x^4 - 60x^3 + 1684x^2 = x^4 - 60x^3 + 1044x^2 - 8640x + 129600$

$640x^2 + 8640x - 129600 = 0$ ÷ 320

$2x^2 + 27x - 405 = 0$

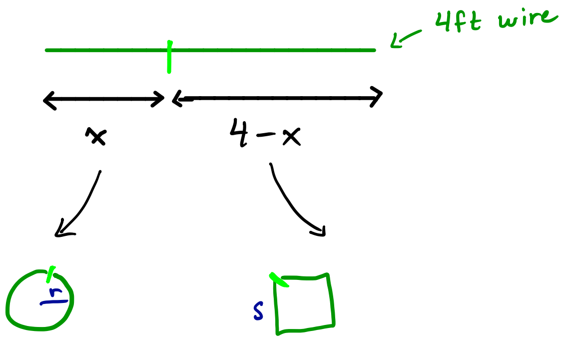
$x = 9$

− 0 + + + ← sign of L′

0 9 30

Ans:
Length is minimized if the wire is fixed 9 ft from the 12ft post.

5. Four feet of wire is used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum area?



circumference is x

So:

$x = 2\pi r$ or

$r = \dfrac{x}{2\pi}$.

Thus

area of circle $= \pi\left(\dfrac{x}{2\pi}\right)^2$

$= \dfrac{x^2}{4\pi}$

perimeter is $4-x$

So:

$4-x = 4s$ or

$s = \dfrac{4-x}{4}$.

Thus,

area square $= \dfrac{1}{16}(4-x)^2$

area enclose $= A(x) =$ area in circle $+$ area in square

$= \dfrac{1}{4\pi}\cdot x^2 + \dfrac{1}{16}(4-x)^2$; domain: $[0,4]$.

Now, $A'(x) = \dfrac{x}{2\pi} + \dfrac{1}{8}(4-x)(-1) = \left(\dfrac{x}{2\pi} - \dfrac{4-x}{8} = 0\right)\cdot 8\pi$

Critical pts: $4x - 4\pi + \pi x = 0$ or $x = \dfrac{4\pi}{4+\pi} \approx 1.76$

$- - - \, 0 \, + + +$ ← sign A'

0   1.76   4

↑ location of local min.

$x=0$, $A(0) = 1$; $x=4$, $A(4) = \dfrac{4}{\pi} > 1$.

So, area is maximized when all of the wire is used to make the circle