RECITATION 12: REVIEW OF CHAPTERS 3 & 4

For this worksheet, you want to focus on more than merely "getting the right answer" especially since you have no reason to expect these particular problems will appear on the midterm. What else should you do while working through these problems?

- Paying attention to writing mathematics correctly.
- Focussing on the general principle being applied – not the particulars of one single problem.

1. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 1$ where the tangent is horizontal.

$\frac{dy}{dx} = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1) = 0$ when $x = 2$ or $x = -1$

Ans: The curve has horizontal tangents at $(-1, 8)$ and $(2, -19)$

2. A particle moves on a vertical line so that it's coordinate at time t is $y = t^3 - 12t + 3$ for $t \geq 0$. Assume t is in seconds and s is in meters.

(a) Find the velocity and the acceleration of the particle at time t.

velocity : $y' = 3t^2 - 12$

acceleration: $y'' = 6t$

(b) When will the velocity be zero?

Set $y' = 0$. So $0 = 3t^2 - 12 = 3(t^2 - 4) = 3(t+2)(t-2)$.

Thus, $t = \pm 2$. (-2 is not in the domain.)

Answer: The velocity is zero at 2 seconds

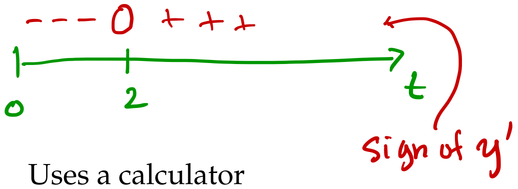
(c) Sketch a diagram of the motion of the particle.

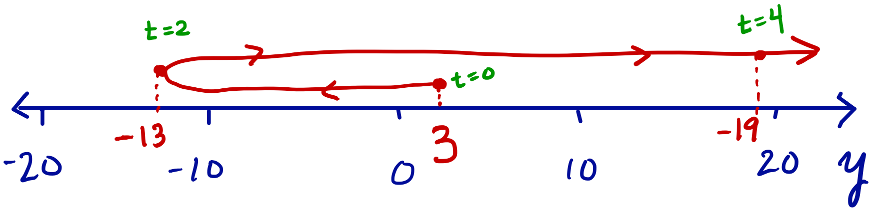
work :

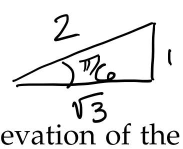
$t = 0$ $y = 3$

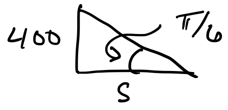
$t = 2$, $y = -13$

From the sign of y', we know the particle moves left, then, at time $t=2$, it turns as goes right.







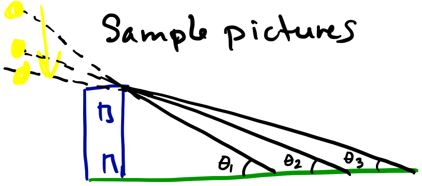


$\frac{400}{s} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt3}$

$s = 400\sqrt3$

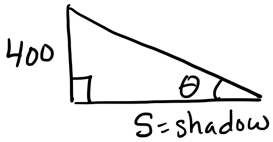
3. The angle of elevation of the sun is decreasing at a rate of 0.25 radian per hour. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is $\pi/6$?

Sample pictures



as time goes on,
$\theta_1 \rightarrow \theta_2 \rightarrow \theta_3$ gets
smaller + shadow
gets longer

general pic:



Find $\frac{ds}{dt}$ when

$s = 400\sqrt3$, $\theta = \pi/6$

and $\frac{d\theta}{dt} = -0.25$

equation: $\tan\theta = \frac{400}{s} = 400\, s^{-1}$

take derivative w.r.t. time t:

$$\sec^2\theta \cdot \frac{d\theta}{dt} = -400 \cdot s^{-2} \cdot \frac{ds}{dt}$$

plug in:

answer

$$\left(\frac{2}{\sqrt3}\right)^2\left(\frac{-1}{4}\right) = \frac{-400}{(400\sqrt3)^2}\cdot\frac{ds}{dt} \;;\; \frac{ds}{dt} = \boxed{400 \text{ ft/hr}}$$

4. The radius of a circular disc is given as 24 cm with a maximum error in measurement of 0.2 cm.

(a) Use differentials to estimate the maximum error in the calculated area of the disk.

$$A = \pi r^2$$

$$dA = 2\pi r\,dr \leq 2\pi(24)(0.2) \approx \underline{30.159 \text{ cm}^2}$$

maximum
error in area

(b) What is the *relative* error? $A = \pi(24)^2$, $\frac{dA}{A} = \frac{2\pi(24)(0.2)}{\pi(24)^2} = 0.01\overline{6}$

5. Find the critical numbers of $f(t) = \frac{\sqrt t}{1+t^2}$.

$$f'(t) = \frac{(1+t^2)\cdot\frac12 t^{-1/2} - t^{1/2}(2t)}{(1+t^2)^2} \cdot \frac{2\sqrt t}{2\sqrt t} = \frac{(1+t^2) - 4t^2}{2\sqrt t\,(1+t^2)^2} = \frac{1-3t^2}{2\sqrt t\,(1+t^2)^2}$$

f' undefined at $t=0$

$f'=0$ at $t = \pm 1/\sqrt3$

Ans: The critical points are

$t = 0, +\frac{1}{\sqrt3}, \frac{-1}{\sqrt3}$

6. (a) Explain how Rolle's Theorem is a special case of the Mean Value Theorem.

Rolle's Thm is the MVThm when $f(a)=f(b)$.

(b) Verify that the function $f(x) = \dfrac{x}{x+2}$ satisfies the two hypotheses of the Mean Value Theorem on the interval $[1, 4]$. Then, find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f'(x) = \frac{(x+2)(1)-x(1)}{(x+2)^2} = 2(x+2)^{-2}.$$

So f and f' are defined for all x in [1,4]. So f(x) is continuous and differentiable in the interval. Thus MVThm applies.

(Now apply it:)

$f(1) = \frac{1}{3}$, $f(4) = \frac{4}{6} = \frac{2}{3}$.

$$\text{So } \frac{f(b)-f(a)}{b-a} = \frac{\frac{2}{3}-\frac{1}{3}}{4-1} = \frac{1}{9}$$

(Find c-values).

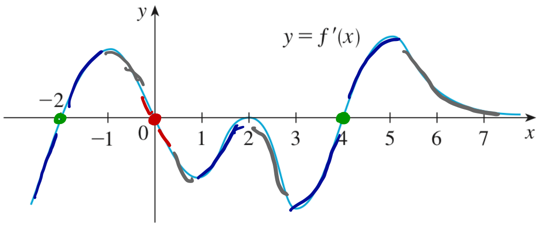
We need c so that $f'(c) = \frac{1}{9}$. That is:

$$\frac{2}{(c+2)^2} = \frac{1}{9} \quad \text{or} \quad \frac{\sqrt{2}}{c+2} = \frac{1}{3}$$

So $3\sqrt{2} = c+2$.

So $c = 3\sqrt{2} - 2 \approx 2.24$ ✓

7. The graph of the first derivative f' of a function f is shown below.



min

max

ccup = f' increasing

cc down = f' decreasin

(a) On what intervals is f increasing?

(want intervals where $f' > 0$)

Ans: $(-2,0) \cup (4,\infty)$

(b) At what values of x does f have a local maximum or minimum?

(+ to −) (− to +)

min at $x=-2, 4$

max at $x=0$

(c) On what intervals is f concave up or concave down?

ccup $(-\infty,-1) \cup (1,2) \cup (3,5)$

cc down $(-1,1) \cup (2,3) \cup (5,\infty)$

(d) What are the x-coordinates of the inflection points of f?

$x = -1, 1, 2, 3, 5$

8. Evaluate the following limits. Show your work.

(a) $$\underbrace{\lim_{x\to\infty} \frac{(\ln x)^2}{e^{2x}}}_{\text{form } \infty/\infty} \overset{H}{=} \lim_{x\to\infty} \frac{2\cdot\ln x\cdot\frac{1}{x}}{2e^{2x}}$$

$$= \underbrace{\lim_{x\to\infty} \frac{\ln x}{x\,e^{2x}}}_{\text{form } \infty/\infty} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{e^{2x}+x\cdot 2e^{2x}}$$

$$= \lim_{x\to\infty} \frac{1}{e^{2x}(x+2x^2)} = 0$$

(b) $$\overbrace{\lim_{x\to 0^+} x\cdot\ln x}^{\text{form } 0\cdot\infty} = \overbrace{\lim_{x\to 0^+} \frac{\ln x}{x^{-1}}}^{\text{form } \frac{\infty}{\infty}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} \cdot \frac{x^2}{x^2}$$

$$= \lim_{x\to 0^+} -\frac{x}{1} = 0$$

(c) $$\underbrace{\lim_{x\to 0^+} x^{\ln x}}_{\text{form } 0^{-\infty} = \frac{1}{0^{\infty}}} = \infty$$

not indeterminate.

(d) $$\overbrace{\lim_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)}^{\text{form } \infty-\infty} \overset{H}{=} \overbrace{\lim_{x\to 1^+} \frac{x\ln x - (x-1)}{(x-1)\ln x}}^{\text{form } \frac{0}{0}}$$

$$= \lim_{x\to 1^+} \frac{1\cdot\ln x + x\cdot\frac{1}{x} - 1}{1\cdot\ln x + (x-1)\cdot\frac{1}{x}} = \lim_{x\to 1^+} \frac{\ln x}{\ln x + 1 - x^{-1}}$$

form $\frac{\infty}{\infty}$

$$\overset{H}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}}{\frac{1}{x} + x^{-2}} = \frac{1}{1+1} = \frac{1}{2}$$

9. Use the Second Derivative Test to identify any local maximums or minimums of $f(x) = 2x(20-x^2)$.

$f(x) = 40x - 2x^3$

$f'(x) = 40 - 6x^2 = 2(20-3x^2) = 0$, So $x = \pm\sqrt{\frac{20}{3}} = \pm 2\sqrt{5}/\sqrt{3}$

$f''(x) = -12x$

$f''(2\sqrt{5}/\sqrt{3}) = -12\left(\frac{2\sqrt{5}}{\sqrt{3}}\right) < 0$ $\cap$. f has a local max at $x = 2\sqrt{5}/\sqrt{3}$.

$f''(-2\sqrt{5}/\sqrt{3}) = -12\left(\frac{-2\sqrt{5}}{\sqrt{3}}\right) > 0$ or $\cup$. So f has a local min at

$x = -2\sqrt{5}/\sqrt{3}$

10. Sketch the graph of $f(x) = \dfrac{1}{x^2-9}$ incorporating the information below.

(a) Find the domain. $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

(b) Find the x and y-intercepts.

if $x=0$, $y=-1/9$ (y-intercept.)

Note $y\neq 0$. So no x-intercept.

(c) Find the symmetries/ periodicity of the curve.

All even terms. $f(x)$ is even.

(d) Determine the asymptotes.

$x=3$, $x=-3$ vertical asymptotes. $\lim\limits_{x\to\infty}\dfrac{1}{x^2-9}=0$. So $y=0$ is a horizontal asymptote.

(e,f) Determine where the function is increasing/ decreasing and find the local maximum/ minimum values

$$f(x) = (x^2-9)^{-1}$$

$$f'(x) = -(x^2-9)^{-2}(2x)$$

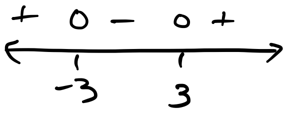
$$= \frac{-2x}{(x^2-9)^2}$$

So $f'>0$ for x in $(-\infty,0)$; $f'<0$ for x in $(0,\infty)$.

ANS: f is increasing on $(-\infty,0)$, and decreasing on $(0,\infty)$.

(g) Find the intervals of concavity/inflection points.

$$f''(x) = \frac{6(x^2+3)}{(x^2-9)^3}$$



f is cc up on $(-\infty,-3)\cup(3,\infty)$ and cc down $(-3,3)$.

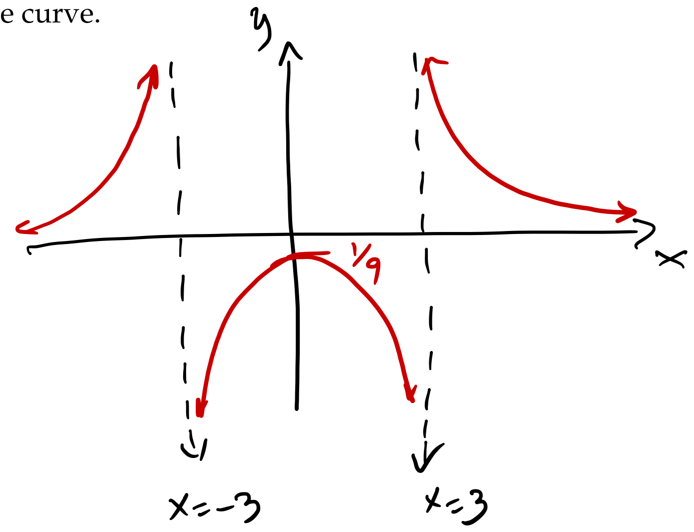
Inflection points : none

$x=-3$ and $x=3$ aren't in the domain.

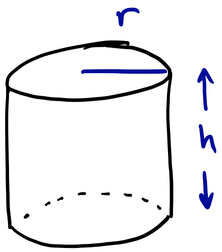
$f''\neq 0$

f'' undefined at $x=\pm 3$

(h) Sketch the curve.



11. A cylindrical can is to be constructed with a volume of 20 cubic feet. The material for the top of the can costs \$2 per square foot. The material for the bottom of the can costs \$10 per square foot. The material used to construct the side of the can costs \$ 5 per square foot. What dimensions of the can will minimize the costs of materials?



$$V = \pi r^2 h = 20. \quad \text{So } h = \frac{20}{\pi r^2} = \underbrace{\frac{20}{\pi} r^{-2}}$$

$$C = \underbrace{2 \cdot \pi r^2}_{\text{cost top}} + \underbrace{10 \cdot \pi r^2}_{\text{cost bottom}} + \underbrace{5 \cdot 2\pi r \boxed{h}}_{\text{cost sides}}$$

$$c(r) = 12\pi r^2 + 10\pi r\left(\frac{20}{\pi} r^{-2}\right) = 12\pi r^2 + 200 r^{-1}$$

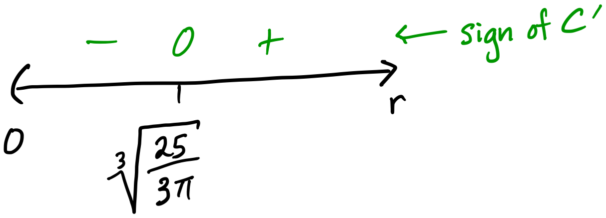
with domain $(0, \infty)$.

$$C'(r) = 24\pi r - 200 r^{-2} = 0$$

So $24\pi r = \frac{200}{r^2}$.

Thus $r^3 = \frac{200}{24\pi} = \frac{25}{3\pi}$.

So $r = \sqrt[3]{25/3\pi}$ is the unique critical point



So, $r = \sqrt[3]{25/3\pi}$ corresponds to an absolute minimum.

So $h = \frac{20}{\pi}\left(\frac{3\pi}{25}\right)^{2/3}$