Recitation: 5-3 The Fundamental Theorem of Calculus (Part 2)

Review: Look back at your notes from class yesterday and state both parts of the Fundamental Theorem of Calculus below.

- **The Fundamental Theorem of Calculus, Part 1**

If f is continuous on $[a,b]$, then the function g defined by $g(x) = \int_a^x f(t)\,dt \quad a \le x \le b$

is cts on $[a,b]$ and diff on (a,b) and $g'(x) = f(x)$

- **The Fundamental Theorem of Calculus, Part 2**

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any anti-derivative of f, as in $F' = f$.

Example 1: Using the FTC (Part 1), differentiate the following functions.

(a) $f(x) = \int_0^x \sqrt{1+t^2}\,dt$

$$\boxed{f'(x) = \sqrt{1+x^2}}$$

(b) $g(x) = \int_{\cos x}^1 (t + \sin t)\,dt$ $= -\int_1^{\cos x} (t + \sin t)\,dt$

$$g'(x) = -(\cos x + \sin(\cos x)) \frac{d}{dx}\cos x$$
$$= -(\cos x + \sin(\cos x))(-\sin x)$$
$$= \boxed{\sin x(\cos x + \sin(\cos x))}$$

Example 2: Using the FTC (Part 1), differentiate $h(x) = \int_{\cos x}^{5x} \cos(u^2)\,du$

$$h(x) = \int_{\cos x}^{a} \cos(u^2)\,du + \int_a^{5x} \cos(u^2)\,du$$
$$= -\int_a^{\cos x} \cos(u^2)\,du + \int_a^{5x} \cos(u^2)\,du$$

$$h'(x) = -\cos(\cos^2 x)(-\sin x) + \cos(5x)^2 \cdot 5$$
$$= \boxed{\sin x \cos(\cos^2 x) + 5\cos(25x^2)}$$

Example 3: Use the second part of the FTC to evaluate the integral, or explain why it does not exist.

(a) $\int_0^4 y(3+2y-y^2)dy = \int_0^4 (3y+2y^2-y^3)dy$

$= \frac{3}{2}y^2 + \frac{2}{3}y^3 - \frac{1}{4}y^4 \Big|_0^4$

$= \frac{3}{2}(16) + \frac{2}{3}(64) - \frac{1}{4}\cdot 4^4$

$= 24 + \frac{128}{3} - 64$

$= 128/3 - 40(3/3) = \boxed{-8/3}$

(b) $\int_1^8 \sqrt[3]{x}dx = \int_1^8 x^{1/3}\,dx$

$= \frac{3}{4}x^{4/3}\Big|_1^8$

$= \frac{3}{4}(8^{4/3} - 1^{4/3})$

$(\sqrt[3]{8})^4$

$= \frac{3}{4}(16-1) = \boxed{45/4}$

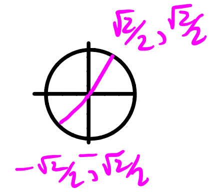
Example 4: Use the second part of the FTC to evaluate the integral, or explain why it does not exist.

(a) $\int_{-1}^7 x^{-3}dx = \int_{-1}^7 \frac{1}{x^3}\,dx$

$\boxed{\text{Does not exist}}$

as $f(x) = 1/x^3$ is not continuous on $[-1,7]$

(b) $\int_{\pi/4}^{5\pi/2} \sin x\,dx = -\cos x\Big|_{\pi/4}^{5\pi/4}$



$= -\cos\frac{5\pi}{4} + \cos \pi/4$

$= -(-\sqrt{2}/2) + \sqrt{2}/2$

$= \boxed{\sqrt{2}}$

Example 5: Use the second part of the FTC to evaluate the integral, or explain why it does not exist.

(a) $\int_1^3 (x-2)(x+3)dx = \int_1^3 (x^2+x-6)\,dx$

$= \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x\right)\Big|_1^3$

$= \left(9 + \frac{9}{2} - 18\right) - \left(1/3 + 1/2 - 6\right)$

$= -9 + 8/2 - 1/3 + 6$

$= -9 + 4 + 6 - 1/3$

$= 1 - 1/3$

$= \boxed{2/3}$

(b) $\int_1^2 \left(x + \frac{1}{x}\right)^2 dx = \int_1^2 \left(x^2 + 2\cdot x\cdot\frac{1}{x} + \frac{1}{x^2}\right)dx$

$= \int_1^2 (x^2 + 2 + x^{-2})\,dx$

$= \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1}\Big|_1^2$

$= \frac{8}{3} + 4 - \frac{1}{2} - \left(1/3 + 2 - 1\right)$

$= 7/3 - 1/2 + 3$

$= \frac{14}{6} - \frac{3}{6} + \frac{18}{6}$

$= \boxed{29/6}$

Example 6: Use the second part of the FTC to evaluate the integral, or explain why it does not exist.

(a) $\int_0^{\pi/4} \sec^2 x dx = \tan x \Big|_0^{\pi/4}$

$= \tan(\pi/4) - \tan 0$

$= \frac{\sin(\pi/4)}{\cos(\pi/4)} - \frac{\sin 0}{\cos 0}$

$= \frac{(\sqrt{2}/2)}{(\sqrt{2}/2)} - \frac{0}{1}$

$= \boxed{1}$

(b) $\int_1^8 \frac{x^2-1}{x} dx = \int_1^8 (x^2-1)x^{-1}\,dx$

$= \int_1^8 (x - 1/x)\,dx$

$= \left(\frac{1}{2}x^2 - \ln|x|\right)\Big|_1^8$

$= \frac{64}{2} - \ln 8 - (1/2 - \ln 1)$

$= \boxed{\frac{63}{2} - \ln 8}$

Example 7: Use the second part of the FTC to evaluate the integral, or explain why it does not exist.

(a) $\int_1^4 \sqrt{2} dx = \sqrt{2}\,x \Big|_1^4$

$= 4\sqrt{2} - 1\sqrt{2}$

$= \boxed{3\sqrt{2}}$

(b) $\int_1^9 \sqrt{\frac{2}{x}} dx = \int_1^9 \frac{\sqrt{2}}{\sqrt{x}}\,dx$

$= \int_1^9 \sqrt{2}\,x^{-1/2}\,dx$

$= \sqrt{2}\,(2\sqrt{x})\Big|_1^9$

$= 2\sqrt{2}(\sqrt{9} - \sqrt{1})$

$= 2\sqrt{2}\,(3-1)$

$= \boxed{4\sqrt{2}}$

Example 8: If $f(1) = 10$, f' is continuous, and $\int_1^5 f'(x)dx = 23$, what is the value of $f(5)$?

$\int_1^5 f'(x)\,dx = f(x)\Big|_1^5$

$23 = f(5) - f(1)$

$23 = f(5) - 10$

$\boxed{f(5) = 33}$

Example 9: Determine whether the statement is true or false. If either case, explain why or give an example that disproves the statement.

(a) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

True!

note we do:

$\int_0^1 (x^2+1)dx = \int_0^1 x^2 dx + \int_0^1 1\, dx$

(b) $\int_a^b [f(x)g(x)]dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$

False!

$\int_0^1 x \cdot x\, dx = \int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = 1/3$

but $\int_0^1 x\, dx \int_0^1 x\, dx = \frac{1}{2}x^2\Big|_0^1 \cdot \frac{1}{2}x^2\Big|_0^1 = \frac{1}{4}$

Example 10: Determine whether the statement is true or false. If it is true, explain why or give an example that disproves the statement.

(a) $\int_a^b \sqrt{f(x)}dx = \sqrt{\int_a^b f(x)dx}$

FALSE!

$\int_0^1 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}$

but $\sqrt{\int_0^1 x\, dx} = \sqrt{\frac{1}{2}x^2\Big|_0^1} = \frac{1}{\sqrt{2}}$

(b) $\int_{-2}^1 \frac{1}{x^4} = -\frac{3}{8}$

FALSE!

note $f(x) = 1/x^4$ has a discontinuity in $[-2, 1]$

Example 11: Evaluate the limits by first recognizing the sum as a Reimann sum for a function.

(a) $\lim_{n\to\infty} \sum_{i=1}^n \overset{\text{width}}{\frac{2}{n}} \Big(\underset{\text{starting point}}{5} + \frac{2i}{n}\Big)^{10}$

$= \int_5^7 x^{10}\, dx$

$= \frac{1}{11}x^{11}\Big|_5^7$

$= \boxed{\frac{1}{11}(7^{11} - 5^{11})}$

(b) $\lim_{n\to\infty} \sum_{i=1}^n \frac{5}{n} \sin\left(2 + \frac{5i}{n}\right)$

$= \int_2^7 \sin(x)\, dx$

$= -\cos(x)\Big|_2^7$

$= -\cos(7) + \cos(2)$

$= \boxed{\cos(2) - \cos(7)}$