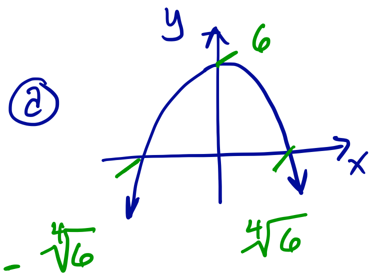
Recitation 5: Midterm I Review

PRACTICE PROBLEMS:

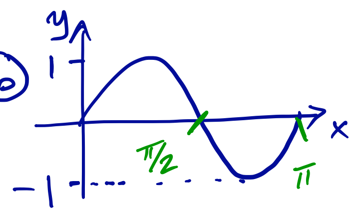
1. Sketch each of the functions below. Label all x- and y-intercepts and asymptotes. State, in interval notation, the domain and range of each function next to its graph.

   (a) $y = 6 - x^4$
   (b) $y = \sin(2x)$
   (c) $y = \tan x$
   (d) $y = \tan^{-1} x$
   (e) $y = e^{x-1} + 2$
   (f) $y = \ln x$
   (g) $y = -2/(x+3)$
   (h) $y = \sqrt{x+5}$

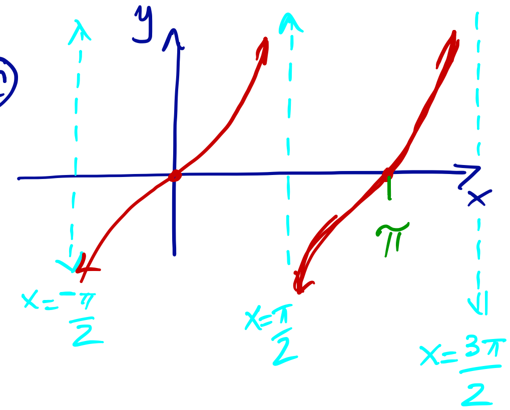
(a)



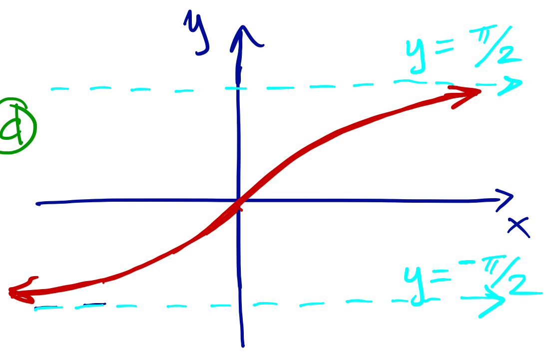
(b)



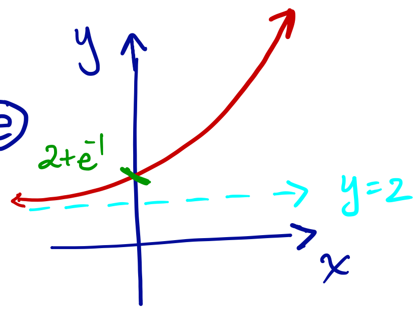
(c)



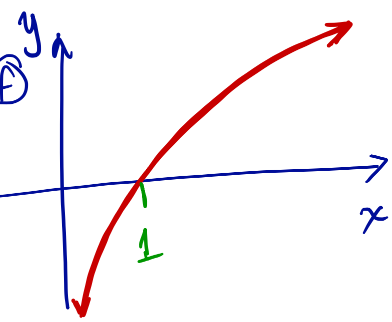
(d)



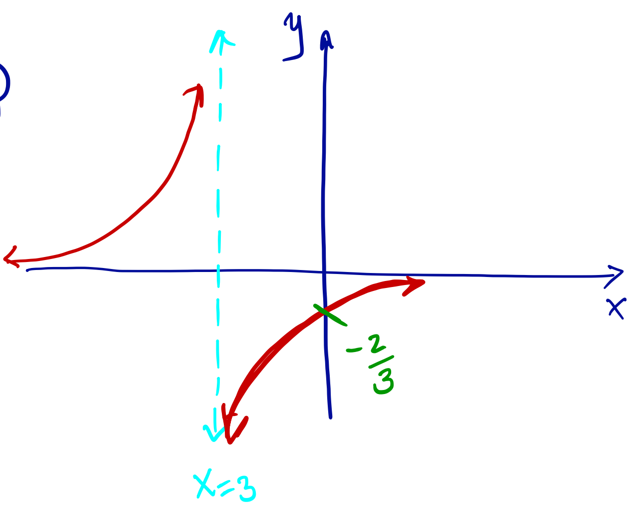
(e)



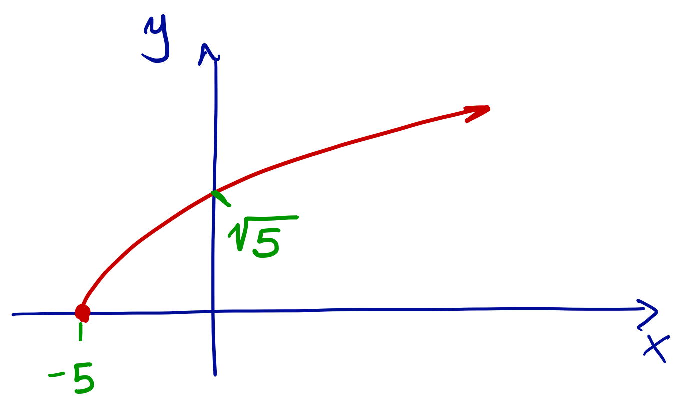
(f)



(g)



(h)



2. Evaluate the following limits. Show your work. *Make sure you are writing your mathematics correctly and clearly.*

(a) $\lim_{t\to 2}\left(\frac{t^2-4}{t^3-3t+5}\right)^3 = \left(\frac{2^2-4}{2^3-3\cdot 2+5}\right)^3 = \left(\frac{0}{7}\right)^3 = 0$

I can just plug in. :)

(b) $\lim_{x\to 4}\frac{x^2+3x}{x^2-x-12} = \lim_{x\to 4}\frac{x(x+3)}{(x+3)(x-4)} = \lim_{x\to 4}\frac{x}{x-4}$ = DNE because

algebra

Note: "lim" still there.

as $x\to 4^+$, $\lim_{x\to 4^+}\frac{x}{x-4}=\infty$ and $\lim_{x\to 4^-}\frac{x}{x-4}=-\infty$

(c) $\lim_{x\to -3}\frac{x^2-4x}{x^2-x-12} = \lim_{x\to -3}\frac{x(x+3)}{(x+3)(x-4)} = \lim_{x\to -3}\frac{x}{x-4} = \frac{-3}{-7} = 3/7$

(d) $\lim_{h\to 0}\frac{(h-5)^2-25}{h} = \lim_{h\to 0}\frac{h^2-10h+25-25}{h} = \lim_{h\to 0}\frac{h(h-10)}{h} = \lim_{h\to 0} h-10 = -10$

3. For each function below, determine all the values in the domain of the function for which the function is continuous. (For the first example, you may find sketching the graph helpful, though you are not required to graph it.)

(a) $f(x)=\begin{cases}\frac{3}{x+5} & x<1\\ \frac{x+1}{2} & 1\le x\le 3\\ x^2-7 & 3<x\end{cases}$

at $x=1$ left: $\frac{3}{6}=\frac{1}{2}$, right: $\frac{2}{2}=1$

at $x=3$ left: $\frac{4}{2}=2$, right: $9-7=2$

also at $x=-5$ as $f(-5)$ is undefined.

Answer: $(-\infty,-5)\cup(-5,1)\cup(1,\infty)$

thinking

concerned about where pieces "meet"

(b) $g(x)=\frac{2^x+1}{\sqrt{1-x}}$

thinking

This will be continuous where it is defined. Only need to check where denominator is zero or undefined.

Need $1-x>0$

So $1>x$

Answer: $(-\infty,1)$

4. Find the limit or show that it does not exist. *Make sure you are writing your mathematics correctly and clearly.*

(a) $\lim_{x\to\infty} \frac{3x^4+2x}{\sqrt{x^4+2x}}$ $\cdot \frac{\frac{1}{x^2}}{1/x^2} = \lim_{x\to\infty} \frac{3x^2 + \frac{2}{x}}{\sqrt{1 + 2/x^3}} = \infty$

(b) $\lim_{x\to\infty} \frac{\cos^2 x}{x^3+1}$ $= 0$ because $0 \le \cos^2 x \le 1$ but $x^3+1 \to \infty$

(c) $\lim_{x\to\infty} \frac{\sqrt[3]{8x^3+1}}{2-5x}$ $\cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{\sqrt[3]{8 + 1/x^3}}{\frac{2}{x} - 5} = \frac{\sqrt[3]{8}}{-5} = \frac{-2}{5}$

5. Write the formula for a function with vertical asymptotes at $x = -1$ and $x = 3$ and a horizontal asymptote at $y = 4/3$.

$$f(x) = \frac{4x^2}{3(x+1)(x-3)}$$

6. State, formally, the definition of the derivative of a function $f(x)$ at $x = a$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$ provided this limit exists.

7. Let $f(x) = 5x^2 - 3x$.

(a) Use the definition to find the derivative of $f(x)$.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{5(x+h)^2 - 3(x+h) - (5x^2-3x)}{h} = \lim_{h\to 0} \frac{5x^2+10xh+5h^2-3x-3h-5x^2+3x}{h}$$

$$= \lim_{h\to 0} \frac{10xh+5h^2-3h}{h} = \lim_{h\to 0} 10x+5h-3 = 10x-3 = f'(x).$$

(b) Find the slope of the tangent line to $f(x)$ when $x = -3$.

$m = \text{slope} = f'(-3) = 10(-3)-3 = -33$

(c) Write the equation of the line tangent to $f(x)$ when $x = -3$.

Find y-value when $x=-3$.

$y = f(-3) = 5(-3)^2 - 3(-3) = 45+9 = 54$.

Point $(-3, 54)$, slope $m=-33$

answer:

$y - 54 = -33(x+3)$ or, equivalently,

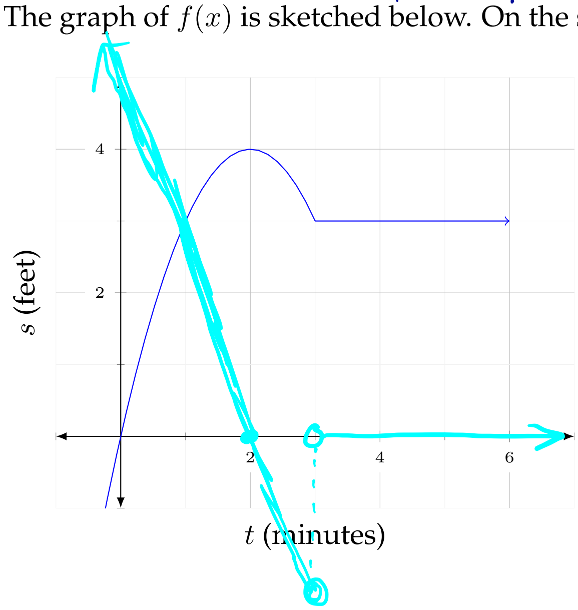
$y = -33x - 45$

8. Suppose N represents the number of people in the United States who travel by car to another state for a vacation this Memorial Day weekend when the average price of gasoline is p dollars per gallon. Would you expect dN/dp to be positive or negative? Explain your answer.

I would expect dN/dp to be negative since if p increases (that is, the price of gas goes up), the number of travellers will (I expect) decrease.

(I am using the idea: $\frac{dN}{dp} \approx \frac{\Delta N}{\Delta p} = \frac{\text{change in } N}{\text{change in } p}$

9. The graph of $f(x)$ is sketched below. On the same set of axes, give a rough sketch $f'(x)$.



thinking

f' is slope.

- from $[3,\infty)$, f is flat so $f'=0$ for these x-values.
- from $(2,3)$, f is decreasing so $f'<0$
- from $(-\infty,2)$, f is increasing so $f'>0$.

10. Find the domain of each function.

(a) $f(x) = \sqrt{x^2 - x - 6}$

need $x^2-x-6 \geq 0$.

$x^2-x-6 = (x-3)(x+2)$

$+ \quad 0 \quad - \quad 0 \quad +$ ← check sign

$-2 \qquad 3$

answer: $(-\infty,-2] \cup [3,\infty)$

(b) $g(t) = \ln(t + 6)$

We need $t+6>0$.

So $t>-6$

answer: $(-6,\infty)$

11. Solve for x.

(a) $e^{x-3} + 2 = 6$

$e^{x-3} = 4$

$x-3 = \ln 4$

$\boxed{x = 3+\ln 4}$

(b) $\ln(x + 5) - 3 = 7$

$\ln(x+5) = 10$

$x+5 = e^{10}$

$\boxed{x = e^{10}-5}$

(c) $\ln x + \ln(x - 1) = 0$

$\ln(x(x-1)) = 0$

$x(x-1) = e^0$

$x^2-x=1$

$x^2-x-1=0$

$x = \dfrac{1 \pm \sqrt{1+4}}{2}$

$\boxed{x = \dfrac{1+\sqrt{5}}{2} \text{ or } \dfrac{1-\sqrt{5}}{2}}$

(d) $\cos(8x + 1) = 0$

$\cos\theta = 0$ if $\theta = \frac{\pi}{2} + \pi k$, k integer.

So $8x+1 = \frac{\pi}{2} + \pi k$

So $x = \frac{\pi}{16} + \frac{\pi k}{8} - 1$, k integer

12. Write the expression below as a single logarithm:

$$\frac{1}{3}\left(\ln(x+2)^3\right) + \frac{2}{3}\left(\ln x - \ln(x^2+1)^2\right)$$

$$= \ln(x+2) + \frac{2}{3}\ln\left(\frac{x}{(x^2+1)^2}\right) = \ln(x+2) + \ln\left[\left(\frac{x}{(x^2+1)^2}\right)^{2/3}\right]$$

$$= \ln\left[(x+2)\left(\frac{x}{(x^2+1)^2}\right)^{2/3}\right] = \ln\left[\frac{x^{2/3}(x+2)}{(x^2+1)^{4/3}}\right]$$

13. Find the exact value of the following expressions.

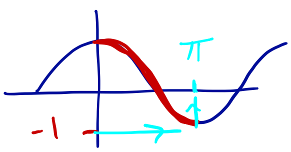
(a) $\cos^{-1}(-1) =$ (b) $\sin^{-1}(0.5) =$ (c) $\tan^{-1}(\sqrt{3}) =$

Need θ so that

$\cos\theta = -1$

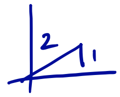
So $\theta = \pi$

(picture:)



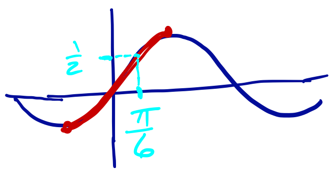
Need θ so that

$\sin\theta = \frac{1}{2}$



So $\theta = \pi/6$

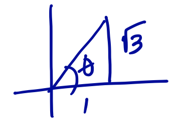
(picture:)



✱

Need θ so that

$\sqrt{3} = \tan\theta$

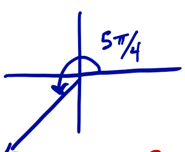


So $\theta = \pi/3$

14. Find the exact value of the following expressions:

(a) $\sin^{-1}(\sin(5\pi/4)) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

work:



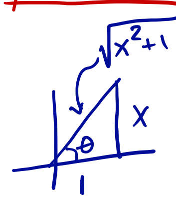
$= -\frac{\pi}{4}$

See ✱ above. We must have θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

(b) $\sin(\tan^{-1} x) = \boxed{\frac{x}{\sqrt{x^2+1}}}$ Space for answer

Let $\theta = \tan^{-1} x$.

So $\tan\theta = x$. So



Now

$\sin(\tan^{-1} x) = \sin(\theta) = \frac{x}{\sqrt{x^2+1}}$

15. Evaluate $\lim_{\theta\to\pi} \sec(\theta + \sin\theta) = \sec\left(\lim_{\theta\to\pi} \theta + \sin\theta\right) = \sec(\pi) = -1$