RECITATION 3
MORE ON SECTION 2-3: CALCULATING LIMITS USING THE LIMIT LAWS

REVIEW: Complete the table below.

Limit Laws

In the rules below c is a constant, n is an integer, and $\lim_{x\to a} f(x)$ and $\lim_{x\to a}(x)$ both exist.

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \left(\lim_{x\to a} f(x)\right) - \left(\lim_{x\to a} g(x)\right)$
3. $\lim_{x\to a}[cf(x)] = c\left(\lim_{x\to a} f(x)\right)$
4. $\lim_{x\to a}[f(x)\cdot g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
5. $\lim_{x\to a}[f(x)/g(x)] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$
12. If $f(x)$ is a polynomial or rational function, then $\lim_{x\to a} f(x) = f(a)$ provided a is in the domain of $f(x)$.
13. The two-sided limit ($\lim_{x\to a} f(x)$) exists if and only if both one-sided limits exist and are equal. $\left(\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)\right)$

GOALS:

- Many limits are evaluated by application of the limit laws above combined with a thoughtful use of *algebra*. We will practice this today.
- There remain limits too slippery for straightforward algebra. For this reason, we will learn a technique for finding limits using a bounding (or "squeezing")approach.
- We will also review the greatest integer function.

Pay attention to HOW you write your solution! Organized? Easy to follow? Correct use of "="? Correct use of "$\lim_{x\to a}$"?

PRACTICE PROBLEMS (SET 1): Evaluate the following limits or explain why they do not exist.

1. $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} = \lim_{x\to -2} \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} = \frac{-1}{5+6} = \boxed{-\frac{1}{11}}$

This is a rational function. I just plug in unless the denominator is zero.

2. $\lim_{h\to 0} \frac{(h-5)^2-25}{h} = \lim_{h\to 0} \frac{(h^2-10h+25)-25}{h} = \lim_{h\to 0} \frac{h^2-10h}{h}$

The denominator is zero when h=0. Must do ALGEBRA prior to plugging in.

$= \lim_{h\to 0} \frac{h(h-10)}{h} = \lim_{h\to 0} h-10 = \boxed{-10}$

why?

It's fair! I haven't changed the problem.

3. (**hint:** rationalize the denominator.) $\lim_{t\to 0} \frac{t}{\sqrt{1+3t}-1} \cdot \frac{\sqrt{1+3t}+1}{\sqrt{1+3t}+1} = \lim_{t\to 0} \frac{t(\sqrt{1+3t}+1)}{1+3t-1}$

$= \lim_{t\to 0} \frac{t(\sqrt{1+3t}+1)}{3t} = \lim_{t\to 0} \frac{\sqrt{1+3t}+1}{3} = \lim_{t\to 0} \frac{\sqrt{1+3\cdot 0}+1}{3}$

$= \boxed{\frac{2}{3}}$

4. $\lim_{z\to 1} \frac{8-z}{c-z}$, where c is a constant.

If $c \neq 1$, then $\lim_{z\to 1} \frac{8-z}{c-z} = \lim_{z\to 1} \frac{8-1}{c-1} = \frac{7}{c-1}$.

If $c=1$, then the limit does not exist. (More specifically, as $z\to 1^+$, $\frac{8-z}{c-z} \to +\infty$. As $z \to 1^-$, $\frac{8-z}{c-z} \to -\infty$.)

5. $\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \lim_{x\to 3} \frac{\frac{3}{3x}-\frac{x}{3x}}{x-3} = \lim_{x\to 3} \frac{3-x}{3x(x-3)} = \lim_{x\to 3} \frac{-(x-3)}{3x(x-3)}$

$= \lim_{x\to 3} \frac{-1}{3x} = \boxed{\frac{-1}{9}}$

6. $\lim_{x\to 0} \frac{\sqrt{16-x}-4}{x} \cdot \frac{\sqrt{16-x}+4}{\sqrt{16-x}+4} = \lim_{x\to 0} \frac{16-x-16}{x(\sqrt{16-x}+4)}$

$= \lim_{x\to 0} \frac{-x}{x(\sqrt{16-x}+4)} = \lim_{x\to 0} \frac{-1}{\sqrt{16-x}+4} = \boxed{\frac{-1}{8}}$

7. $\lim_{x\to 2} \frac{x^2-4}{2x^2-3x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(2x+1)} = \lim_{x\to 2} \frac{x+2}{2x+1} = \boxed{\frac{4}{5}}$

thinking:
$2\cdot 2^2-3\cdot 2-2=8-8=0$.
So $x=2$ is a root of $2x^2-3x-2$. So it factors w/ a term $x-2$.

8. $\lim_{x\to 0} \frac{1}{x}-\frac{1}{|x|}$ = DNE because the left-hand limit and right-hand limit are not equal.

my answer

thinking:

my explanation

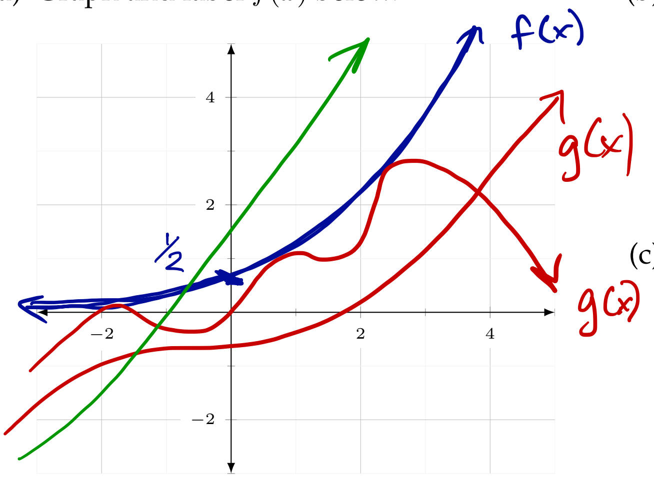
As $x\to 0^+$, $|x|=x$. So $\lim_{x\to 0^+} \frac{1}{x}-\frac{1}{|x|} = \lim_{x\to 0^+} \frac{1}{x}-\frac{1}{x} = \lim_{x\to 0^+} 0 = 0$

As $x\to 0^-$, $|x|=-x$. So $\lim_{x\to 0^-} \frac{1}{x}-\frac{1}{|x|} = \lim_{x\to 0^-} \frac{1}{x}+\frac{1}{x} = \lim_{x\to 0^-} \frac{2}{x} = -\infty$

PRACTICE PROBLEMS (SET 2): The Squeeze Theorem

1. Let $f(x) = \frac{1}{2}e^x$ and assume $g(x)$ is a function with the property that $g(x) \le f(x)$ for all real numbers.

   (a) Graph and label $f(x)$ below.



   (b) While we do not know exactly what $g(x)$ looks like, graph two *different* possible graphs of $g(x)$ and one example of a graph that could not possibly be $g(x)$. Compare your pictures with your neighbors' drawings.

   (c) Assume $\lim_{x\to 1} g(x) = L$, what can you say (if anything) about L?

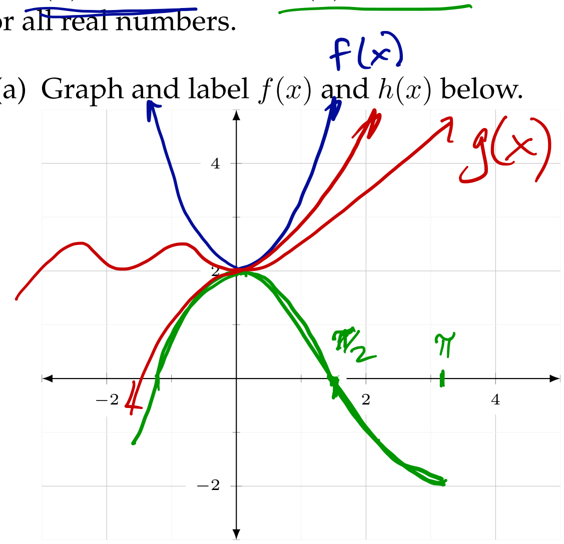
   $$\lim_{x\to 1} g(x) \le \lim_{x\to 1} \tfrac{1}{2}e^x = \tfrac{1}{2}e^1 = \frac{e}{2}$$

   So $L \le \frac{e}{2}$.

   That is, whatever L is... it can't be more than $\frac{e}{2}$.

2. Let $f(x) = x^2 + 2$ and let $h(x) = 2\cos x$. Assume that $g(x)$ is a function such that $h(x) \le g(x) \le f(x)$ for all real numbers.

   (a) Graph and label $f(x)$ and $h(x)$ below.



   (b) While we do not know exactly what $g(x)$ looks like, graph two *different* possible graphs of $g(x)$ and one example of a graph that could not possibly be $g(x)$. Compare your pictures with your neighbors' drawings.

   (c) What (if anything) can you say about $\lim_{x\to 0} g(x)$?

   $$\lim_{x\to 0} g(x) = \lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 2$$

   (d) What (if anything) can you say about $\lim_{x\to \pi} g(x)$?

   $$-2 \le \lim_{x\to \pi} g(x) \le \pi^2 +$$

3. Fill in the blanks in the formal statement of the Squeeze Theorem:

If $h(x) \le g(x) \le f(x)$ $\boxed{\text{for all real numbers}}$ and $\lim_{x\to a} h(x) = L = \lim_{x\to a} f(x)$,

then $\boxed{\lim_{x\to a} g(x) = L}$.

That is, $g(x)$ gets squeezed in between the two functions $f(x)$ and $h(x)$.

Note: It is sufficient for the inequality to hold close to a.

4. Explain why you cannot evaluate $\lim_{\theta \to 0} \theta^2 \sin\left(\frac{1}{\theta}\right)$ by plugging in zero?

You get 0 in the denominator of the term $\frac{1}{\theta}$.

5. Use the Squeeze Theorem to evaluate the limit $\lim_{\theta \to 0} \theta^2 \sin\left(\frac{1}{\theta}\right)$.

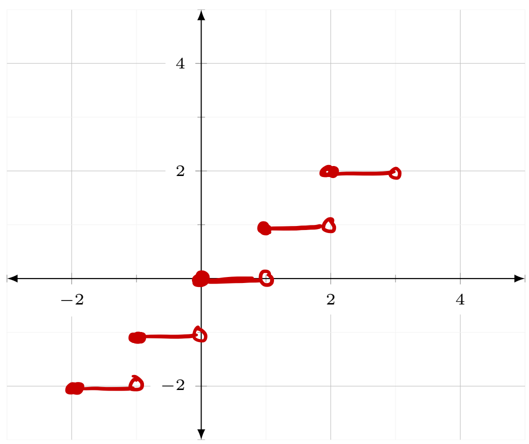
We need to pick an $f(x)$ and an $h(x)$. We can use the fact about the sine function: $-1 \le \sin\left(\frac{1}{\theta}\right) \le 1$. Multiply this inequality by θ^2 to get: $-\theta^2 \le \theta^2 \sin\left(\frac{1}{\theta}\right) \le \theta^2$. Since $\lim_{\theta \to 0} -\theta^2 = 0 = \lim_{\theta \to 0} \theta^2$, we can conclude $\lim_{\theta \to 0} \theta^2 \sin\left(\frac{1}{\theta}\right) = 0$. (That is, we squeezed $\theta^2 \sin\left(\frac{1}{\theta}\right)$ between $-\theta^2$ and θ^2.)

PRACTICE PROBLEMS (SET 3): The Greatest Integer Function

Recall that the function $f(x) = [\![x]\!]$ is called *the greatest integer function* and outputs the greatest integer less than or equal to the input x. (Note that in other contexts this function is sometimes called *the floor function.*

1. Fill out the chart:

x	-2	-1	0	0.5	0.75	0.999	1	1.1	1.5	1.999	2
$[\![x]\!]$	-2	-1	0	0	0	0	1	1	1	1	2



$f(x) = [\![x]\!] = \lfloor x \rfloor$

floor function

2. Sketch the graph of $f(x) = [\![x]\!]$:

3. Evaluate the limits below, if possible. If not, explain why they do not exist. Let n be an arbitrary integer.

(a) $\lim_{x \to 5^+} [\![x]\!] = 5$

(b) $\lim_{x \to 5^-} [\![x]\!] = 4$

(c) $\lim_{x \to 5} [\![x]\!] = \text{DNE}$.
LH limit is not equal RH limit.

(d) $\lim_{x \to 5.5} [\![x]\!] = 5$

(e) $\lim_{x \to 5.5} (3[\![x]\!] + \sqrt{2}) = 3 \cdot 5 + \sqrt{2} = 15 + \sqrt{2}$

(f) $\lim_{x \to n^+} [\![x]\!] = n$

(g) $\lim_{x \to n^-} [\![x]\!] = n-1$