§ 3.10

linearization, $L(x)$, of $f(x)$ at $x=a$	means	tangent line to graph of $f(x)$ at $x=a$ with y replaced by $L(x)$

<u>Example</u>. Find the linearization of $f(x)=\sqrt{x}$ at $a=4$.

<u>work</u>: $f(4)=\sqrt{4}=2$ point $(4,2)$

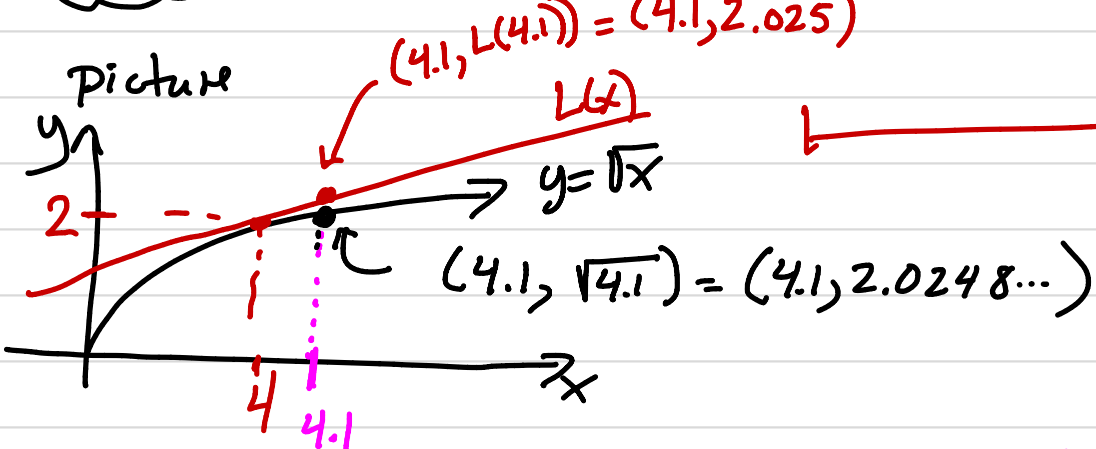
$$f'(x)=\frac{1}{2}x^{-1/2}, \quad m=f'(4)=\frac{1}{2}(4)^{-1/2}=\frac{1}{4}$$

tangent line: $y-2=\frac{1}{4}(x-4)$ or $y=2+\frac{1}{4}(x-4)$

<u>ANSWER</u>: $\boxed{L(x)}=2+\frac{1}{4}(x-4)$ note: I don't simplify...

Why? $L(x)$ is good at approximating $f(x)$ NEAR $a=4$.

Picture



Look at problem 1

The point is that these y-values (output values) are REALLY close!

1. Use the linear approximation of $f(x) = \sqrt{x}$ at $x = 4$ to approximate $\sqrt{4.1}$ and compare your result to its approximation computed by your calculator.

$\frac{1}{4}\cdot(0.1) = (0.25)(0.1) = 0.025$

$\to L(x) = 2 + \frac{1}{4}(x-4)$ (from previous page)

- Plug in $x = 4.1$ into $L(x)$: $L(4.1) = 2 + \frac{1}{4}(4.1-4) = 2 + \frac{1}{4}\cdot(0.1) = 2.025$

THIS is our approximation.

use calculator.

- Compare: $f(4.1) = \sqrt{4.1} = 2.0248\ldots$
  to $L(4.1) = 2.025$

error ≈ 0.0002 (!!)

← See picture on previous page!!

2. Use the linear approximation to approximate the cosine of $29^\circ = \frac{29}{30}\frac{\pi}{6}$ radians.

What is $f(x)$? a?

- $f(x) = \cos x$
- $a = 30^\circ$

Work: Find tangent line

$f(\pi/6) = \sqrt{3}/2$

$f'(\pi/6) = -\sin(\pi/6) = -\frac{1}{2}$

① linearization

$L(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)$

approximation

$L\left(\frac{29}{30}\cdot\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{29}{30}\frac{\pi}{6} - \frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{1}{30} = \frac{30\sqrt{3}+1}{60} = \boxed{0.87475..}$

answer

aside: $\cos(29^\circ) = 0.874619$

3. Find the linear approximation of $f(x) = \ln(x)$ at $a = 1$ and use it to approxmate $\ln(0.5)$ and $\ln(0.9)$. Compare your approximation with your calculator's. Sketch both the curve $y = \ln(x)$ and $y = L(x)$ and label the points $A = (0.5, \ln(0.5))$ and $B = (0.5, L(0.5))$

- $f(1) = \ln(1) = 0$. point $(1, 0)$

$f'(x) = \frac{1}{x}$, $f'(1) = \frac{1}{1} = 1 = m$

line: $y - 0 = 1(x-1)$

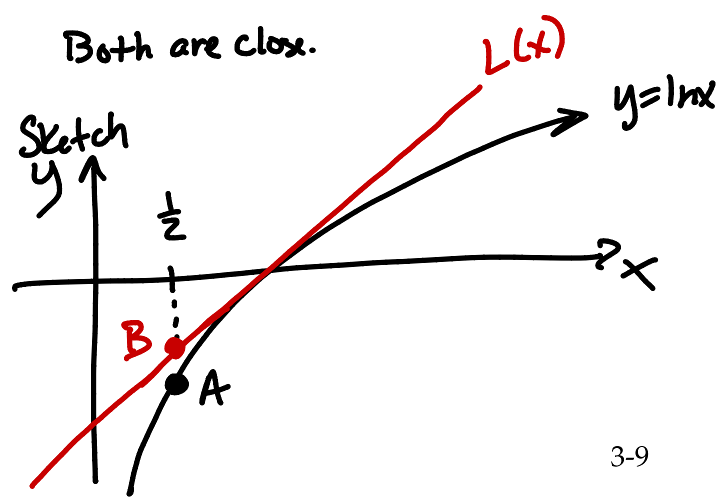
$\boxed{L(x) = x - 1}$

- $L(0.5) = 0.5 - 1 = -0.5$
  $L(0.9) = 0.9 - 1 = -0.1$

- Compare $\ln(0.5) = -0.6931$ vs. -0.5
  $\ln(0.9) = -0.105$ vs. -0.1

Both are close.

- Sketch



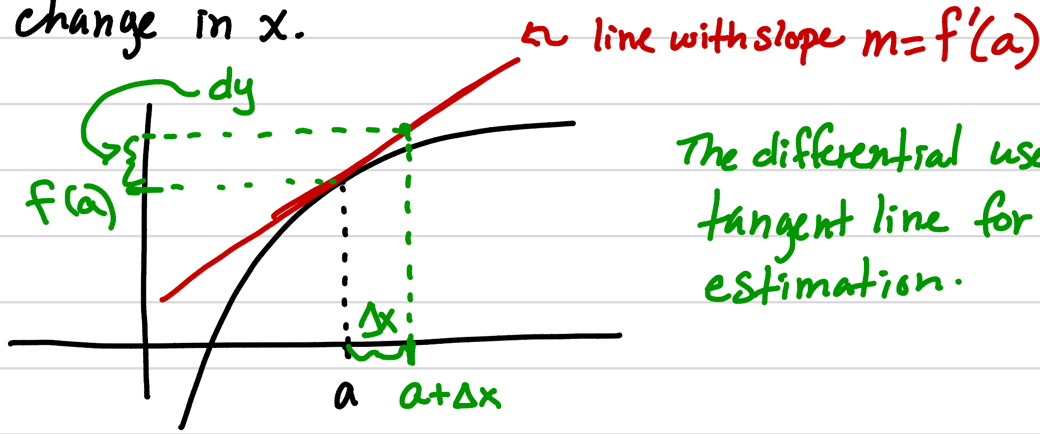
Differential

$\boxed{\text{the differential of } y=f(x)}$ means $\boxed{dy = f'(x)\,dx}$

Example: The differential of $y = x^{\frac{1}{2}}$ is $\boxed{dy = \frac{1}{2}x^{-\frac{1}{2}}\,dx}$

Why? The differential estimates how much y changes for a given change in x.

Picture



The differential uses the tangent line for this estimation.

look at problem 4 →

4. A tree is growing and the radius of its trunk in centemeters is $r(t) = 2\sqrt{t}$ where t is measured in years. Use the differential to estimate the change in radius of the tree from 4 years to 4 years and one month.

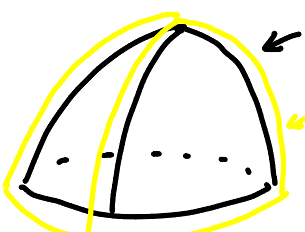
- the differential: $dr = 2 \cdot \frac{1}{2} \cdot t^{-1/2}\, dt$ or $\boxed{dr = \frac{1}{\sqrt{t}}\, dt}$ ← the differential

- estimate Δr as t goes from 4 years to $4\frac{1}{12}$ year.

So $\Delta t = \frac{1}{12}$. Plug in: $t=4$, $dt = \frac{1}{12}$ to get

$$dr = \frac{1}{\sqrt{4}} \cdot \frac{1}{12} = \frac{1}{24} \text{ cm}$$

5. A coat of paint of thinkness 0.05cm is being added to a hemispherical dome of radius 25m. Estimate the volume of paint needed to accomplish this task. [Challenge: will this be an underestimate or an overestimate? Thinking geometrically or thinking algebraically will both give you the same answer.]



← hemisphere

paint

0.05 cm thick paint

function? Volume: $V = \frac{4}{3}\pi r^3 \left(\frac{1}{2}\right)$ ← half-Sphere

$V = \frac{2}{3}\pi r^3$ → differential

$$\boxed{dV = 2\pi r^2\, dr}$$

- Estimation: $r = 25\text{m}$, $dr = 0.05\text{cm} = \frac{0.05}{100}\text{m} = 0.0005\text{m}$

So $dV = \frac{2\pi (25)^2\, 0.05}{100} = 1.96 \text{ m}^3$

6. The radius of a disc is 24cm with an error of ± 0.5cm. Estimate the error in the area of the disc as an absolute and as a relative error.

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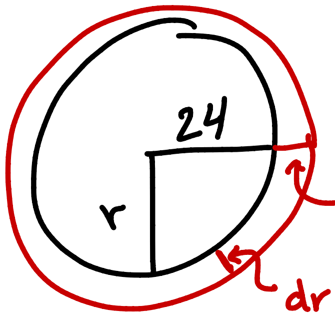

$A = \pi r^2$

$dA = 2\pi r\, dr$

$dA = 2\pi \cdot 24 \cdot \frac{1}{2} = 24\pi \approx 75.4\text{cm}$

absolute error.

For a $\frac{1}{2}$ cm error in measurement of r, a calculation of area A can have an error of as much as 75.4 cm.

$$\text{relative error} = \frac{dA}{A} = \frac{75.4}{\pi(25)^2} = 0.0384 = 3.84\%$$

So an error of $\frac{1}{2}$ cm can produce an error in calculated area by as much as 3.84%.