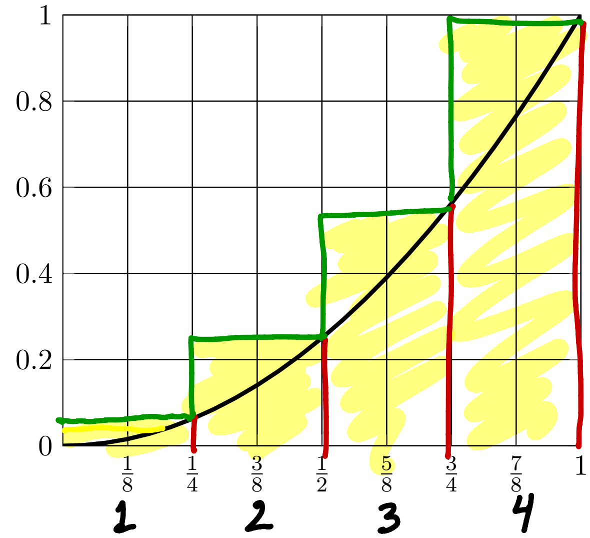
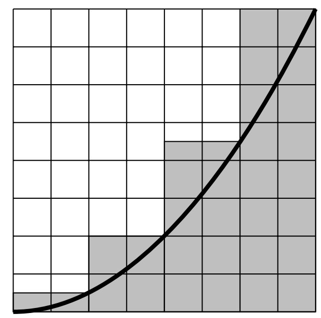
Approximating the area under a curve

Below is a piece of the graph of $y = x^2$. We wish to estimate the area under the curve bounded between the x-axis, $x = 0$, $x = 1$, and the curve.





1. Slice up the axis between $x = 0$ and $x = 1$ into 4 evenly spaced slices. Use the right-hand edge of the slice to construct rectangles whose height is the height of the function at the right-hand edge (i.e., you are using the RIGHT-HAND ENDPOINT of the slice to construct the rectangle). Actually draw the rectangles on the above graph! Your picture should look something like the right hand smaller figure. Compute the area of each rectangle: **Make sure to use the height of the rectangle determined from the function:** don't estimate it from the graph! (For example, the height of the first rectangle is $(1/4)^2 = 1/16$)

   Area of rectangle 1: $b \cdot h = \frac{1}{4} f(\frac{1}{4}) = \frac{1}{4} \cdot \frac{1}{16} = \frac{1}{64}$

   Area of rectangle 2: $\frac{1}{4} \cdot f(\frac{1}{2}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

   Area of rectangle 3: $\frac{1}{4} \cdot f(\frac{3}{4}) = \frac{1}{4} \cdot \frac{9}{16} = \frac{9}{64}$

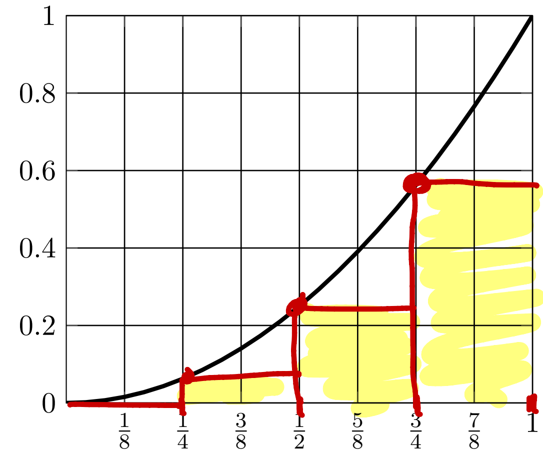
   Area of rectangle 4: $\frac{1}{4} \cdot f(1) = 1 = \frac{1}{4}$

   Total area of the rectangles: $\frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4} = \frac{30}{64} = \frac{15}{32}$

   Is your estimated area bigger or smaller than the total area under the curve? Why?

   bigger. The rectangles are above the curve.

2. Now slice up the area under the curve into four equally-spaced slices. Draw in 4 rectangles and use the LEFT-HAND ENDPOINT of each slice to determine the height of the rectangle. Compute the area of each rectangle.

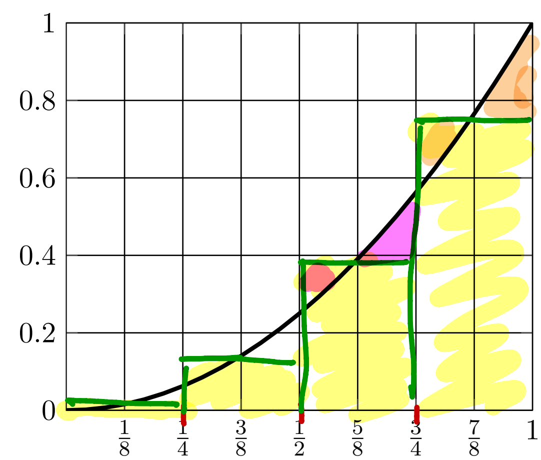


Determine the total area of the rectangles (use the function to determine the height, not the graph!):

$$A = \frac{1}{4}\cdot f(0) + \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) = \frac{1}{4}\left(0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right) = \frac{14}{64} = \frac{7}{32}$$

Is this an overestimation or an underestimation? Why? underestimate

3. Now slice up the segment between $x = 0$ and $x = 1$ into four equal pieces. Using the MIDPOINT of each piece to determine the height of each rectangle, draw in four rectangles, and determine the area of each rectangle (use the function to determine the heights, not the picture!).



Determine the total area of the rectangles (use the function to determine the heights, not the picture!):

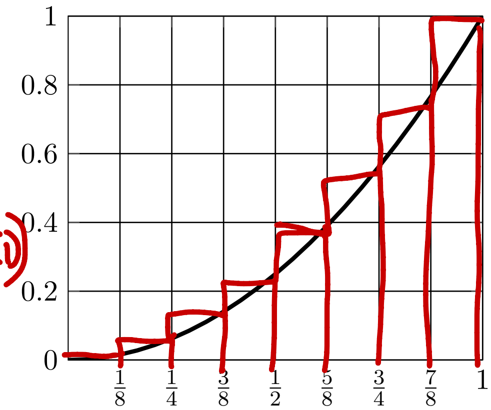
$$A = \frac{1}{4} f\left(\frac{1}{8}\right) + \frac{1}{4} f\left(\frac{3}{8}\right) + \frac{1}{4} f\left(\frac{5}{8}\right) + \frac{1}{4} f\left(\frac{7}{8}\right) = \frac{1}{4}\left(\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right) = \frac{1}{4}\,\frac{84}{64} = \frac{21}{64}$$

What do you think: based on your rectangles, is this an overestimation or an underestimation? (The other two rectangle types should have been obvious; this one is considerably more subtle. Make some sort of argument in favor of one choice or the other.)

Underestimate.

Better area approximations

4. Now we want to try to get a better estimation for the area. This time draw in 8 rectangles, using the RIGHT-HAND ENDPOINT to construct each rectangle.



$A = \frac{1}{8}\left(f(\frac{1}{8}) + f(\frac{1}{4}) + f(\frac{3}{8}) + f(\frac{1}{2}) + f(\frac{5}{8}) + f(\frac{3}{4}) + f(\frac{7}{8}) + f(1)\right)$

$= \frac{1}{8}\left(\frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4} + \frac{25}{64} + \frac{9}{16} + \frac{49}{64} + 1\right)$

Write down a calculation (you don't have to actually do the computation) to determine the area of the 8 rectangles. Is this area a more accurate estimation of the area under the curve than when you used 4 rectangles? Why?

more acurate

5. Suppose the odometer on our car is broken and we want to estimate the distance driven over a 1.5 hour time period. We take speedometer readings every 15 minutes and then record them in the table below. Estimate the distance traveled by the car. What method are you using?

Time (minutes)	0	15	30	45	60	75	90
Velocity (mi/h)	17	21	24	29	32	31	28

(use RHS): $D = \frac{1}{4}(21) + \frac{1}{4}(24) + \frac{1}{4}(29) + \frac{1}{4}(32) + \frac{1}{4}(31) + \frac{1}{4}(28) = 41.25$ miles

(use LHS): $D = \frac{1}{4}(17+21+24+29+32+31) = 38.5$ miles

6. Oil leaked out of a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at 2 hour time intervals are shown in the table. Estimate how much oil leaked out. What method are you using? Is it an overestimate or an underestimate.

t (h)	0	2	4	6	8	10
r(t) (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

RHS: $L = 2(7.6+6.8+6.2+5.7+5.3) = 63.2$ l

LHS: $L = 2(8.7+7.6+6.8+6.2+5.7+5.3) = 70$ l