SECTION 5-3: THE FUNDAMENTAL THEOREM OF CALCULUS

Example 1: If f is the function whose graph is shown and $g(x) = \int_0^x f(t)dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, $g(5)$, and $g(6)$. Then, sketch a rough graph of g.

(a) $g(0) =$ ____________________

(b) $g(1) =$ ____________________

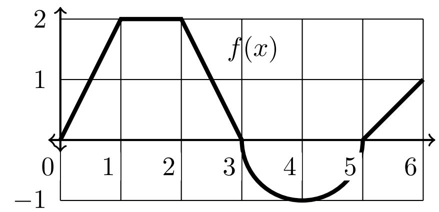
(c) $g(2) =$ ____________________

(d) $g(3) =$ ____________________

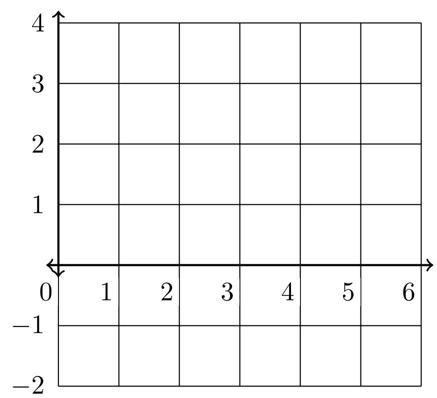
(e) $g(4) =$ ____________________

(f) $g(5) =$ ____________________

(g) $g(6) =$ ____________________



Sketch of $g(x)$



(i) Where is $g(x)$ increasing? ____________________

(ii) Describe f when $g(x)$ is increasing. ____________________

(iii) Where is $g(x)$ decreasing? ____________________

(iv) Describe f when $g(x)$ is decreasing. ____________________

(v) Where does $g(x)$ have a local maximum? ____________________

(vi) Describe f when $g(x)$ has a local max. ____________________

(vii) Where does $g(x)$ have a local minimum? ____________________

(viii) Describe f when $g(x)$ has a local min. ____________________

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, the function g defined by

$$g(x) = \int_a^x f(t)\, dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$.

Example 2: Find the derivative of $g(x) = \displaystyle\int_2^x t^2\, dt$.

Example 3: The Fresnel function $S(x) = \int_0^x \sin(\pi t^2/2)dt$ first appeared in Fresnel's theory of the diffraction of light waves. Recently it was be applied to the design of highways. Find the derivative of the Fresnel function.

Example 4: Find the derivative of the following functions. (Hint: we need to use the chain rule! For part (a), let $u = x^4$...)

(a) $g(x) = \int_1^{x^4} \sec t dt$

(b) $g(x) = \int_{2x+1}^{2} \sqrt{t} dt$

Example 5: Find the derivative of $g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}} dt$. (Hint: we only know the derivative of $\int_a^x f(t)\, dt$, so you need to break this into pieces...)

The Fundamental Theorem of Calculus (Part 2) If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is **any antiderivative** of f, that is, is a function such that $F' = f$. To evaluate, we write $\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a)$.

Example 6: Evaluate the following integrals.

(a) $\int_0^1 x^2\, dx$

(b) $\int_1^4 (1 + 3y - y^2)\, dy$

To compute integrals effectively you **must** have your basic antidifferentiation formulas down. You should know that antiderivatives to the following functions. Note, we are using the $\int$ symbol to mean "find the antiderivative" of the function right after the symbol.

Antiderivatives of Common Functions:

- $\int x^n \, dx =$ ________________
- $\int \sin x \, dx =$ ________________
- $\int \cos x \, dx =$ ________________
- $\int \sec^2 x \, dx =$ ________________
- $\int \sec x \tan x \, dx =$ ________________
- $\int \csc^2 x \, dx =$ ________________
- $\int \csc x \cot x \, dx =$ ________________
- $\int e^x \, dx =$ ________________
- $\int a^x \, dx =$ ________________
- $\int \frac{1}{1+x^2} \, dx =$ ________________
- $\int \frac{1}{\sqrt{1-x^2}} \, dx =$ ________________
- $\int \frac{1}{x} \, dx =$ ________________

Example 7: Evaluate the following integrals.

(a) $\displaystyle\int_2^5 \frac{3}{x} \, dx$

(b) $\displaystyle\int_0^{\pi/2} \cos x \, dx$

Example 8: Evaluate the following integrals.

(a) $\displaystyle\int_1^8 \sqrt[3]{x} \, dx$

(b) $\displaystyle\int_{\pi/6}^{\pi/2} \csc x \cot x \, dx$

(c) $\displaystyle\int_0^1 \frac{9}{1+x^2} \, dx$

Example 9: We do not have any product or quotient rules for antidifferentiation. To evaluate an integral that is expressed as a product or quotient you must try to manipulate the integrand (the stuff inside the $\int$ sign) to look like something you know how to antidifferentiate. The following integrals are examples of this. Evaluate the following integrals.

(a) $\displaystyle\int_1^3 \frac{x^3 + 3x^6}{x^4}\, dx$

(b) $\displaystyle\int_0^1 x(3 + \sqrt{x})\, dx$

Example 10: Evaluate the following integrals.

(a) $\displaystyle\int_0^2 (5^x + x^5)\, dx$

(b) $\displaystyle\int_{1/2}^{\sqrt{2}/2} \frac{1}{\sqrt{1 - x^2}}\, dx$

Example 11: What is wrong with the following calculation?

$$\int_{-1}^3 \frac{1}{x^2}\, dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^3 = -\frac{1}{3} - 1 = -\frac{4}{3}$$