# SECTION 4.4 INDETERMINATE FORMS AND L'HOSPITAL'S RULE

WARM UP: Consider the limit $\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6}$.

If you plug in 2 for $x$, what does the limit "look like"? $\frac{4-4}{4-10+6} = \frac{0}{0}$

Evaluate the limit, using algebraic techniques from Chapter 2, and justifying each step.

$$\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x-3)}$$

$$= \lim_{x\to 2} \frac{x+2}{x-3}$$ ← Since $\frac{(x-2)(x+2)}{(x-2)(x-3)}$ and $\frac{x+2}{x-3}$ differ only at $x=2$

$$= \frac{2+2}{2+3}$$

$$= \frac{4}{-1} = \boxed{-4}$$

A limit is of *indeterminate type* (or 'type' for short) if we can't just "plug in $a$" to find the limit, or if different ways to write the same expression produce different limits! We describe indeterminate types by evaluating the limit of "pieces" by "plugging in $a$" and writing the resulting symbols, for example, $\frac{\infty}{\infty}$, $\frac{0}{0}$, or $\infty - \infty$. We can use *L'Hopital's rule* to help evaluate certain limits of indeterminate type.

**L'Hospital's Rule:** If a limit has the form (indeterminate type) of $0/0$ or $\infty/\infty$,

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} \overset{L'H}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

*provided*

$\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists or is $\pm\infty$.

1. Determine whether or not l'Hospital's Rule applies to the following examples, and if it does, apply it, and determine the limit.

a) $\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6}$ (type $0/0$)

$$\overset{L'H}{=} \lim_{x\to 2} \frac{2x}{2x-5}$$

$$= \frac{4}{4-5} = \boxed{-4}$$ ← which matches what we found above!

b) $\lim_{x\to 0} \frac{\sin x}{x}$ (type $\frac{0}{0}$)

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos(x)}{1}$$

$$= \cos(0) = \boxed{1}$$

QUESTION Why does l'Hospital's Rule work?

Idea: If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is type $\frac{0}{0}$ then $f(a) = 0 = f(b)$, so

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - \overbrace{f(a)}^{=0}}{g(x) - \underbrace{g(a)}_{=0}} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} \cdot \frac{x-a}{x-a} = \lim_{x\to a} \frac{\overbrace{\frac{f(x)-f(a)}{x-a}}^{\to f'(a)}}{\underbrace{\frac{g(x)-g(a)}{x-a}}_{\to g'(a)}}$$

*Evaluate the following limits. Use L'Hopital's rule only when necessary!*

2. $\lim_{x\to 0} \frac{\tan(5x)}{\sin(3x)}$ (type $\frac{0}{0}$)

L'H

$$= \lim_{x\to 0} \frac{\sec^2(5x)(5)}{\cos(3x)(3)}$$

$$= \lim_{x\to 0} \frac{5}{3} \cdot \frac{1}{\cos^2(5x)\cos(3x)}$$

$$= \boxed{5/3}$$

4. $\lim_{x\to 0} \frac{\cos(4x)}{e^{2x}}$ (type $1/1$)

$$= \frac{\cos(4\cdot 0)}{e^{2\cdot 0}}$$ ← we can just evaluate the limit!

$$= \boxed{1}$$

Suppose we'd tried to use L'H?

$$\lim_{x\to 0} \frac{-\sin(4x)(4)}{e^{2x}\cdot 2} = -2 \lim_{x\to 0} \frac{\sin(4x)}{e^{2x}}$$

$$= -2 \cdot \frac{0}{1} = 0$$ ← which is NOT the limit!

Warning: If you try to use L'H when it doesn't apply, it can give you garbage!

3. $\lim_{u\to\infty} \frac{e^{u/10}}{u^2}$ (type $\frac{\infty}{\infty}$)

L'H

$$= \lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{2u} \quad \text{type } \frac{\infty}{\infty}$$

L'H

$$= \lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}\cdot\frac{1}{10}}{2}$$

$$= \frac{1}{200} \lim_{u\to\infty} e^{u/10}$$

$$= \boxed{\infty}$$

5. $\lim_{x\to 0} \frac{xe^x}{2^x - 1}$ (type $0/0$)

L'H

$$= \lim_{x\to 0} \frac{xe^x + e^x}{2^x \ln(2)}$$

$$= \frac{0\cdot e^0 + e^0}{2^0 \cdot \ln(2)}$$

$$= \frac{0+1}{\ln(2)}$$

$$= \boxed{\ln(2)}$$

## Trickier applications of L'Hopital's rule

| Indeterminate form | technique | NOT indeterminate forms | limit |
|---|---|---|---|
| $\frac{0}{0}$ | Algebra; L'H if necessary | $\infty + \infty$ | $\infty$ |
| $\frac{\infty}{\infty}$ | Algebra; L'H if necessary | $1^0$ | $1$ |
| $\infty - \infty$ | algebra to rewrite as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ | $\frac{1}{\infty}$ | $0$ |
| $0 \cdot \infty$ | algebra to rewrite as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ | $\infty \cdot \infty$ and $\infty^\infty$ | $\infty$ |
| $1^\infty$ | Use logs to transform | $\frac{1}{0}$ | $\infty$ |
| $0^0$ | Use logs to transform | $0^\infty$ | $0$ |
| $\infty^0$ | Use logs | $\infty^\infty$ | $\infty$ |

*Transform the following expressions into a form where you can use L'Hopital's rule to evaluate the limit, and then evaluate the limit.*

6. $\lim\limits_{x\to 1^+} \left(\ln(x^4-1) - \ln(x^9-1)\right)$ (type $\infty - \infty$) (ok, $-\infty + \infty$, which is the same!)

$$= \lim_{x\to 1^+} \ln\left(\frac{x^4-1}{x^9-1}\right)$$

$$= \ln\left(\lim_{x\to 1^+} \frac{x^4-1}{x^9-1}\right) \quad \text{type } \frac{0}{0}$$

L'H

$$= \ln\left(\lim_{x\to 1^+} \frac{4x^3}{9x^8}\right)$$

$$= \ln\left(\lim_{x\to 1^+} \frac{4}{9}\cdot\frac{1}{x^5}\right)$$

$$= \boxed{\ln\left(4/9\right)}$$

7. $\lim\limits_{x\to\infty} \sqrt{x}e^{-x/2}$ (type $\infty \cdot 0$)

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x/2}} \quad \text{type } \frac{\infty}{\infty}$$

L'H

$$= \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{e^{x/2}\cdot\frac{1}{2}}$$

$$= \lim_{x\to\infty} \frac{1}{4\sqrt{x}\,e^{x/2}} \longrightarrow \text{gets really big}$$

$$= \boxed{0}$$

**Using logarithms to deal with limits of the form $1^\infty$ or $0^0$**

8. Simplify the expressions below:

(a) If $y = a^b$, then $\ln y =$ $\ln(a^b) = b\ln(a)$ ~~(write your answer using ln(□))~~.

(b) If $\lim_{x\to a} \ln[f(x)] = L$, then $\lim_{x\to a} f(x) =$ $e^L$. write $e^x = \exp(x)$.

why? If $\lim_{x\to a} \ln(f(x)) = L$, then $\exp\left(\lim_{x\to a} \ln(f(x))\right) = \exp(L)$

$\Rightarrow \lim_{x\to a} \exp(\ln(f(x))) = \exp(L) \Rightarrow \lim_{x\to a} f(x) = e^L$.

9. Now find the limit of the functions below by first taking the natural logarithm of the expression in the limit (like part(a) above). Then evaluate the limit of this transformed expression. Finally, use the answer of the transformed expression to obtain the limit of the original expression (like part (b) above).

(a) $\lim_{x\to\infty} x^{2/x}$ (type $\infty^0$)

$y = x^{2/x} \Rightarrow \ln(y) = \ln(x^{2/x}) = \frac{2}{x}\ln(x) = \frac{2\ln(x)}{x}$, so

$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{2\ln(x)}{x}$ type $\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{2\cdot 1/x}{1} = \lim_{x\to\infty} \frac{2}{x} = 0.$

So $\lim_{x\to\infty} y = \lim_{x\to\infty} x^{2/x} = e^0 = \boxed{1}$

(b) $\lim_{x\to 0^+} (1+\sin(2x))^{1/x}$ (type $1^0$)

Let $y = (1+\sin(2x))^{1/x}$. Then $\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \frac{1}{x}\ln(1+\sin(2x))$

$= \lim_{x\to 0^+} \frac{\ln(1+\sin(2x))}{x}$ type $\frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{2\cos(2x)}{1+\sin(2x)}}{1}$

$= \frac{2(1)}{1+0} = 2.$ Therefore $\lim_{x\to 0^+} (1+\sin(2x))^{1/x} = \boxed{e^2}$