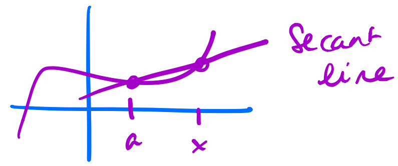
1. Complete the definition: The derivative of a function f at $x = a$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ ← limit of slopes of secant lines.



2. Consider the function $f(x) = 1/3x^2$, shown in the graph below.

   (a) Find the slope of the tangent line to $f(x)$ at $x = a$ by taking the limit of the slopes of secant lines. When you are done, check whether or not your solutions seems plausible, by sketching tangent lines at the marked points and determining the slopes of the tangent lines at those points using your calculation.

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{\frac{1}{3}x^2 - \frac{1}{3}a^2}{x-a} = \lim_{x \to a} \frac{\frac{1}{3}(x-a)(x+a)}{x-a}$$

$$= \frac{1}{3} \lim_{x \to a} x+a = \frac{1}{3}(2a) = \frac{2a}{3}$$

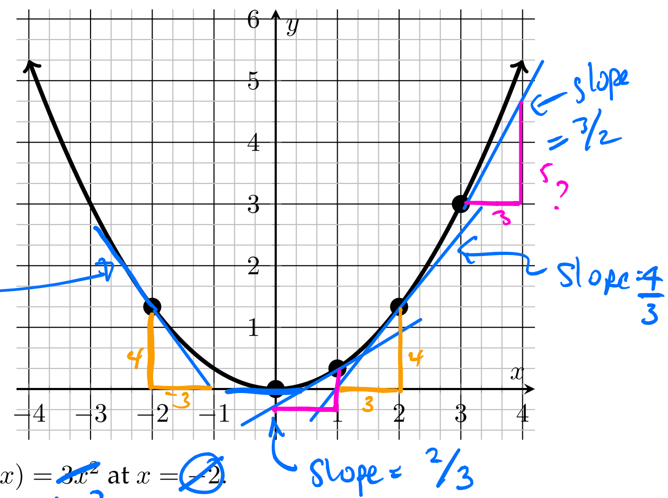
at $a = 0$, $\frac{a}{3} = 0$

$a = 1$, $2a/3 = 2/3$

$a = 2$, $2a/3 = 4/3$

$a = -2$, $2a/3 = -4/3$

$a = 3$, $2a/3 = 2$



   (b) Write the equation of the line tangent to the graph of $f(x) = \cancel{3x^2}$ $\frac{1}{3}x^2$ at $x = \cancel{-2}$ $+1$.

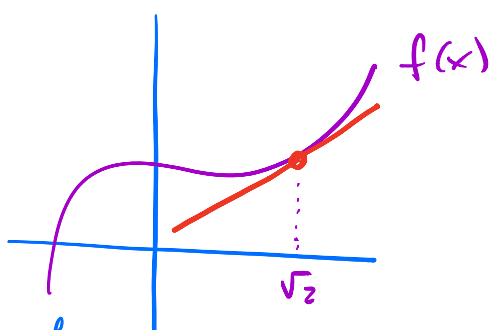
Slope = $-4/3$, so TL is

$$y = \frac{2}{3}(x-1) + \frac{1}{3}$$

3. Assume the tangent line to the graph of $y = f(x)$ at $x = \sqrt{2}$ has equation $y = \frac{4}{3}x - \frac{1}{3}$. Determine:

   (a) $f(\sqrt{2})$

Observe TL and function agree at $x = \sqrt{2}$. So $f(\sqrt{2}) =$ TL at $\sqrt{2}$. Thus $f(\sqrt{2}) = \frac{4}{3}(\sqrt{2}) - 1/3$.



   (b) $f'(\sqrt{2})$

$f'(\sqrt{2})$ measures the slope of the TL at $\sqrt{2}$, which is $\frac{4}{3}$. So $f'(\sqrt{2}) = \frac{4}{3}$.

4. The height in meters of an object is given by the function $s(t) = \frac{2t}{t+1}$ where t is measured in seconds.

   (a) Find $s'(a)$ using the definition in # 1 on this sheet.

   (b) Determine the units of $s'(a)$.

   (c) Find and interpret in the context of the problem the meaning of $s'(1)$.

(a) $$S'(a) = \lim_{t \to a} \frac{S(t) - S(a)}{t-a} = \lim_{t \to a} \frac{\frac{2t}{t+1} - \left(\frac{2a}{a+1}\right)}{t-a}$$

$$= \lim_{t \to a} \frac{2t(a+1) - 2a(t+1)}{(t+1)(a+1)(t-a)} = \lim_{t \to a} \frac{\cancel{2ta} + 2t - \cancel{2ta} - 2a}{(t+1)(a+1)(t-a)}$$

$$= \lim_{t \to a} \frac{2\cancel{(t-a)}}{(t+1)(a+1)\cancel{(t-a)}} = \frac{2}{(a+1)^2}$$

(b) Units are meters/second

(c) $S'(1) = \frac{2}{2^2} = \frac{1}{2}$. So $S'(1)$ means that after one second, the height of the object is increasing at a rate of $\frac{1}{2}$ m/s.

5. Let $f(t) = \sqrt{90 - t}$

   (a) Find $f'(a)$ using the definition in # 1 on this sheet.

   (b) If f is measured in degrees Celsius and t is measured in minutes, determine the units of $f'(a)$.

   (c) Find and interpret $s'(0)$.

(a) $$f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t-a} = \lim_{t \to a} \frac{(\sqrt{90-t}) - (\sqrt{90-a})}{t-a}$$

$$= \lim_{t \to a} \left(\frac{(\sqrt{90-t}) - (\sqrt{90-a})}{t-a} \right) \left(\frac{\sqrt{90-t} + \sqrt{90-a}}{\sqrt{90-t} + \sqrt{90-a}} \right) = \lim_{t \to a} \frac{\cancel{90} - t - \cancel{90} + a}{(t-a)(\sqrt{90-t} + \sqrt{90-a})}$$

$= -t+a$

$$= \lim_{t \to a} \frac{-\cancel{(t-a)}}{\cancel{(t-a)}(\sqrt{90-t} + \sqrt{90-a})} = \lim_{t \to a} \frac{-1}{\sqrt{90-t} + \sqrt{90-a}} = \frac{-1}{2\sqrt{90-a}}$$

(b) Units of $f'(a)$ are °C/minutes

(c) $S'(0) = \frac{-1}{2\sqrt{90}}$. At time = 0, the temperature is decreasing at a rate of $1/2\sqrt{90}$ °C per minute.