1. Use the linear approximation of $f(x) = \sqrt{x}$ at $x = 4$ to approxmiate $\sqrt{4.1}$ and compare your result to its approximation computed by your calculator.

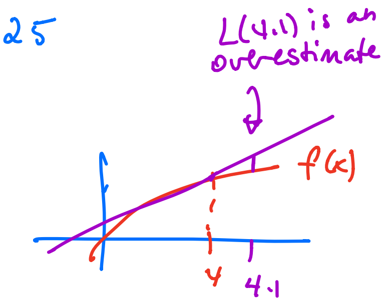
The linearization of $f(x)$ at $x=4$: $f'(x) = \frac{1}{2}x^{-1/2}$ so

$f'(4) = \frac{1}{2\sqrt{4}} = 1/4$, and thus $L(x) = \frac{1}{4}(x-4) + 2$

So $L(4.1) = \frac{1}{4}\left(4 + \frac{1}{10} - 4\right) + 2 = 2 + \frac{1}{40} = 2.025$

By calculator, $f(4.1) = 2.02484$

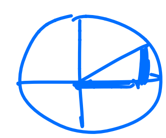
$f(4.1) - L(4.1) = -0.00015$ not too bad



2. Use the linear approximation to approximate the cosine of $29° = \frac{29}{30}\frac{\pi}{6}$ radians.

We will use $L(x)$ at $x = \pi/6$ since we know $\cos(\pi/6)$ exactly.

$\frac{d}{dx}(\cos(x)) = -\sin(x)$ so $\left.\frac{d}{dx}\right|_{x=\pi/6} = -\sin(\pi/6) = -1/2$



Thus $L(x) = -\frac{1}{2}(x - \pi/6) + \frac{\sqrt{3}}{2}$, so

$L(29°) = L\left(\frac{29}{30}\cdot\frac{\pi}{6}\right) = -\frac{1}{2}\left(\frac{29}{30}\cdot\frac{\pi}{6} - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(\frac{-1}{30}\right) + \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{2} + \frac{1}{60}$

3. Find the linear approximation of $f(x) = \ln(x)$ at $a = 1$ and use it to approximate $\ln(0.5)$ and $\ln(0.9)$. Compare your approximation with your calculator's. Sketch both the curve $y = \ln(x)$ and $y = L(x)$ and label the points $A = (0.5, \ln(0.5))$ and $B = (0.5, L(0.5))$

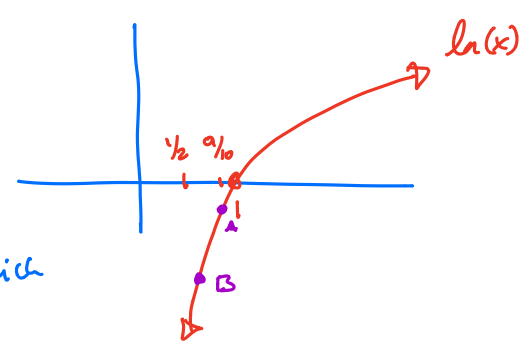
Note $\ln(1) = 0$ and $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ so $\left.\frac{df}{dx}\right|_{a=1} = 1.$

Thus the linearization of $f(x)$ at $a=1$ is

$L(x) = 1(x-1) + 0 = x - 1.$

$L(0.5) = L(1/2) = -1/2$ $\quad \ln(1/2) = -.69314$

$L(0.9) = L\left(\frac{9}{10}\right) = -1/10$ $\quad \ln(.9) = -0.10536$



Tangent line approx is off by almost .2 at $a = 1/2$; not great. However, approx at 9/10 is accurate to 0.005, which is not bad.

4. A tree is growing and the radius of its trunk in centimeters is $r(t) = 2\sqrt{t}$ where t is measured in years. Use the differential to estimate the change in radius of the tree from 4 years to 4 years and one month.

Differential: idea is $\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} \Rightarrow \Delta y \approx \frac{dy}{dt} \Delta t$.

Our Δt = 1 month = $\frac{1}{12}$ years, so

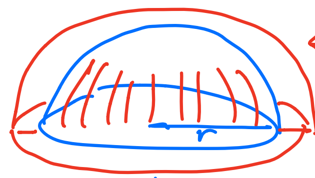
$\Delta y \approx r'(4) \cdot \frac{1}{12}$. Note $r'(t) = 2 \cdot \frac{1}{2} t^{-1/2} = \frac{1}{\sqrt{t}}$ so

$r'(4) = \frac{1}{2}$. Therefore, $\Delta y \approx \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24}$

$\longrightarrow$ we estimate the tree grew about $1/24$ cm in the month

5. A coat of paint of thickness 0.05cm is being added to a hemispherical dome of radius 25m. Estimate the volume of paint needed to accomplish this task. [Challenge: will this be an underestimate or an overestimate? Thinking geometrically or thinking algebraically will both give you the same answer.]

We are estimating the change in volume.



We know $V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$

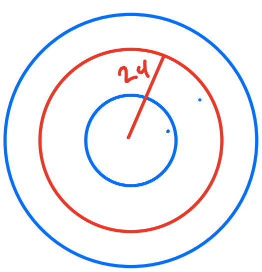
and $\frac{\Delta V}{\Delta r} \approx \frac{dV}{dr} \Rightarrow \Delta V \approx \frac{dV}{dr} \cdot \Delta r$. We know $\Delta r = (0.05 \text{ cm})\left(\frac{1 \text{ m}}{100 \text{ cm}}\right) = \frac{5}{100} \cdot \frac{1}{100} = \frac{5}{10000} = .00005$ m

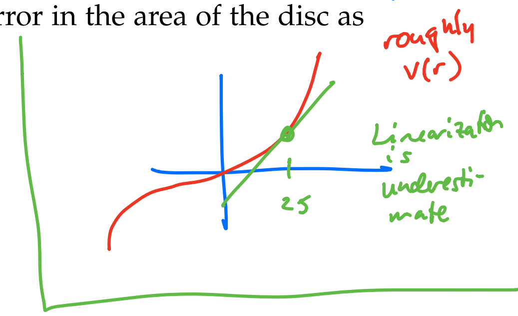
$\frac{dV}{dr} = \frac{2}{\cancel{3}}\pi(\cancel{3}r^2)$ so when $r = 25$, $\frac{dV}{dr} = 2\pi(25)^2$.

Therefore, $\Delta V \approx \cancel{2} \cdot \pi \cdot \cancel{25} \cdot \cancel{25} \cdot \frac{5}{\cancel{100}_{\cancel{4}\,2} \cdot \cancel{100}_{4}} = \frac{5\pi}{8}$ cubic meters $\approx 1.96\ m^3$

≈ 1960 Liters

6. The radius of a disc is 24cm with an error of ± 0.5cm. Estimate the error in the area of the disc as an absolute and as a relative error.





$\Delta A \approx \frac{dA}{dr} \cdot \Delta r$

Know $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

So $\Delta A \approx 2\pi(24)(\pm 0.5)$

abs. error: $\Delta A = \pm 48\pi(0.5) = \pm 24\pi$ Total area $= \pi(24)^2$

relative error $= \frac{\Delta A}{A} = \frac{\pm 24\pi}{\pi(24)^2} = \pm\frac{1}{24} = \pm 0.0416\ldots$ so relative error is $\pm 4\frac{1}{6}$ %