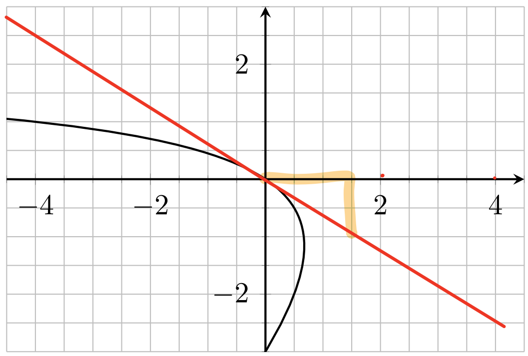
Section 3.5 Implicit Differentiation

1. Find $\frac{dy}{dx}$ for $2x + 3y = xy - y^2$ and find the equations of tangents to the graph when $x = 0$. Use the portion of the curve shown below as an aid and to determine the plausibility of your answers.



(Estimate: $m = -\frac{2}{3}$)

$$\frac{d}{dx}(2x+3y) = \frac{d}{dx}(xy - y^2) \Rightarrow$$

$$2 + 3\frac{d}{dx}(y) = x\frac{d}{dx}(y) + y\frac{d}{dx}(x) - 2y\frac{dy}{dx} \Rightarrow$$

$$2 + 3y' = xy' + y(1) - 2yy' \Rightarrow$$

$$3y' - xy' + 2yy' = y - 2 \Rightarrow$$

$$y'(3 - x + 2y) = y - 2 \Rightarrow$$

$$y' = \frac{y-2}{3-x+2y}$$

So $y'\Big|_{(0,0)} = \frac{0-2}{3-0+2(0)} = \frac{-2}{3}$ is the slope of the TL at $(0,0)$

2. Find $\dfrac{da}{db}$ for $a^3\sin(3b) = a^2 - b^2$. (Pay attention here: b is the independent variable (like x) and a is the dependent variable (like y).

$$\frac{d}{db}(a^3\sin(3b)) = \frac{d}{db}(a^2 - b^2) \Rightarrow$$

$$a^3 \cdot \frac{d}{db}(\sin(3b)) + \sin(3b)\frac{d}{db}(a^3) = \frac{d}{db}(a^2) - \frac{d}{db}(b^2) \Rightarrow$$

$$a^3(\cos(3b)(3)) + \sin(3b)\left(3a^2\frac{da}{db}\right) = 2a\frac{da}{db} - 2b \Rightarrow$$

$$\frac{da}{db}(\sin(3b)\cdot 3a^2 - 2a) = -2b - 3a^3\cos(3b) \Rightarrow \boxed{\frac{da}{db} = \frac{-2b - 3a^3\cos(3b)}{-2a + 3a^2\sin(3b)}}$$

3. Find $\frac{dy}{dx}$ for $e^{xy} = x + y + 1$

$$\frac{d}{dx}(e^{xy}) = \frac{d}{dx}(x+y+1) \Rightarrow$$

$$e^{xy}\frac{d}{dx}(xy) = 1 + \frac{dy}{dx} + 0 \Rightarrow$$

$$e^{xy}\left[x\frac{dy}{dx} + y(1)\right] = 1 + \frac{dy}{dx} \Rightarrow$$

$$\frac{dy}{dx}(xe^{xy}) - \frac{dy}{dx} = 1 - ye^{xy} \Rightarrow$$

$$\frac{dy}{dx}(xe^{xy} - 1) = 1 - ye^{xy} \Rightarrow$$

$$\frac{dy}{dx} = \frac{1 - ye^{xy}}{xe^{xy} - 1}$$

$$= \frac{1 - ye^{xy}}{-1 + xe^{xy}}$$

4. You are going to derive the formula for the derivative of inverse tangent the way we found the derivative of inverse sine in the video.

   (a) Find dy/dx for the expression $x = \tan(y)$.

   $$\frac{d}{dx}(x) = \frac{d}{dx}(\tan(y)) \Rightarrow 1 = (\sec(y))^2 \frac{dy}{dx} \Rightarrow$$

   $$\frac{dy}{dx} = \frac{1}{(\sec(y))^2}$$

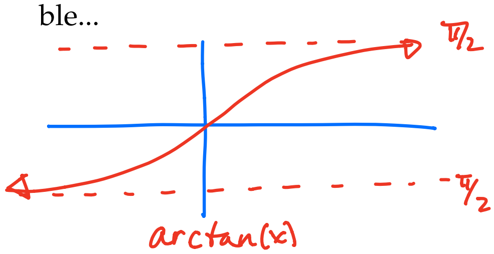
   (b) Use the identity $1 + (\tan(\theta))^2 = (\sec(\theta))^2$ to rewrite you answer in part (a) and *write your dy/dx in terms of x only*.

   $$\frac{dy}{dx} = \frac{1}{(\sec(y))^2} = \frac{1}{1 + (\tan y)^2}$$ But $\tan(y) = x$, so

   $$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

   (c) Now fill in the blank $\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$ because if $x = \tan y$, then $y = \arctan(x)$

   (d) Use your knowledge of the *graph* of $f(x) = \arctan(x)$ to decide if your answer seems plausible...

   - Observe that the slopes of TL are positive on $(-\infty, \infty)$ and $\frac{1}{1+x^2} > 0 \ \forall x$ ✓
   - Observe $\lim_{x \to \pm\infty}$ (slopes of TL) $= 0$ and $\lim_{x \to \pm\infty} \frac{1}{1+x^2} = \lim_{x \to \pm\infty} \frac{1/x^2}{1/x^2 + 1} = 0$ ✓

5. Find the derivative of $f(x) = x \arctan x$.

$$\frac{d}{dx}(x \arctan(x)) = x \frac{d}{dx}(\arctan x) + \arctan(x)\,(1)$$

$$= x\left(\frac{1}{1+x^2}\right) + \arctan(x)$$

6. Find the derivative of $f(x) = \arctan(4 - x^2)$.

$$f'(x) = \left(\frac{1}{1 + (4-x^2)^2}\right)(-2x) = \frac{-2x}{1+(4-x^2)^2}$$

* Note: This sort of thing will end up being very useful in Calculus II!