SECTION 3.6 PART 2: MORE LOGARITHMIC DIFFERENTIATION
SECTION 3.7 PART 1: RATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES

1. We can use laws of logarithms and implicit differentiation to differentiate functions that are overly complicated/impossible to differentiate with the rules we already have.

   (a) Find the derivative of $y = (3x - x^5)^{2/3}(x - \tan(x))^5$.

   i. First, take the natural log of both sides, use laws of logarithms to change the product into a sum and pull down the exponents.

$$\ln(y) = \ln\left[(3x-x^5)^{2/3}(x-\tan(x))^5\right]$$
$$= \ln\left((3x-x^5)^{2/3}\right) + \ln\left((x-\tan(x))^5\right)$$
$$= \tfrac{2}{3}\ln(3x-x^5) + 5\ln(x-\tan(x))$$

   ii. Next, implicitly differentiate everything with respect to x.

$$\frac{y'}{y} = \frac{2(3-5x^4)}{3(3x-x^5)} + \frac{5(1-(\sec(x))^2)}{x-\tan(x)}$$

   iii. Solve for $\frac{dy}{dx}$ and substitute what y equals (from the original function definition) in for y to find the derivative!

$$y' = \left[\frac{2(3-5x^4)}{3(3x-x^5)} + \frac{5(1-(\sec(x))^2)}{x-\tan(x)}\right]\left[(3x-x^5)^{2/3}(x-\tan(x))^5\right]$$
$$= \frac{2(3-5x^4)(x-\tan(x))^5}{3(3x-x^5)^{1/3}} + 5(1-(\sec(x))^2)(3x-x^5)^{2/3}(x-\tan(x))^4$$

Which is what you would have gotten with the product rule followed by the chain rule.

   (b) Find the derivative of $y = (\sin(x))^x$.

   i. First, take the natural log of both sides, and use laws of logarithms to pull the x down from the exponent.

$$\ln(y) = \ln(\sin(x)^x) = x\ln(\sin(x))$$

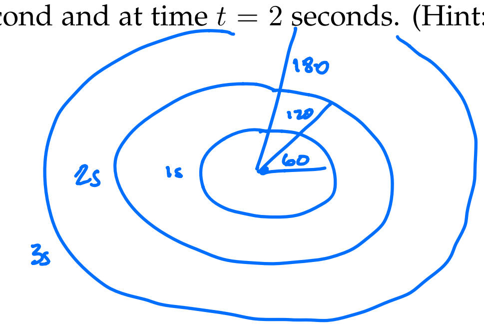
   ii. Next, implicitly differentiate everything with respect to x.

$$\frac{y'}{y} = x \cdot \frac{\cos(x)}{\sin(x)} + \ln(\sin(x))$$

   iii. Solve for $\frac{dy}{dx}$ and substitute what y equals (from the original function definition) in for y to find the derivative!

$$y' = (x\cot(x) + \ln(\sin(x))) \cdot (\sin(x))^x$$

2. A stone is thrown in a pond and a circular ripple travels outward at a speed of 60 cm/s. Determine the rate of change of area inside the ripple at time $t = 1$ second and at time $t = 2$ seconds. (Hint: draw a picture!)



radius at time $t = r(t) = 60t$

A = area at time t = $\pi(r(t))^2 = \pi(60t)^2$

$$\frac{dA}{dt} = 2\pi(60t)\cdot 60 = 7200\pi t$$

$$\left.\frac{dA}{dt}\right|_{t=1} = 7200\pi \text{ cm/s} \qquad \left.\frac{dA}{dt}\right|_{t=2} = 14400\pi \text{ cm/s}$$

$60 \times 60 = 3600$, $\times 2 = 7200$

$7200 \times 2 = 14400$

3. A population of caribou is growing, and its population is

$$P(t) = 4000\frac{3e^{t/5}}{1+2e^{t/5}}.$$

(a) What is the population at time $t = 0$?

$P(0) = 4000$ caribou

(b) Determine the rate of change of the population at any time t.

$$\frac{dP}{dt} = 4000\left(\frac{(1+2e^{t/5})(3e^{t/5}\cdot\frac{1}{5}) - (3e^{t/5})(2e^{t/5}\cdot\frac{1}{5})}{(1+2e^{t/5})^2}\right) \text{ caribou/year}$$

(c) Determine the rate of change of the population at time $t = 0$ years.

$$\left.\frac{dP}{dt}\right|_{t=0} = 4000\left(\frac{(1+2)(3/5) - (3)(2/5)}{(1+2)^2}\right) = 4000\left(\frac{9/5 - 6/5}{9}\right)$$

$$= 4000\left(\frac{3/5}{9}\right) = 4000\left(\frac{3}{5}\cdot\frac{1}{9}\right) = \frac{4000}{15} = \frac{5(800)}{5(3)} = \frac{800}{3} = 266\tfrac{2}{3}$$

Increase of 266 caribou/year at year 0

(d) Determine the long term population.

$$\lim_{t\to\infty} P(t) = 4000 \lim_{t\to\infty} \frac{3e^{t/5}}{1+2e^{t/5}} = 4000 \lim_{t\to\infty} \frac{3}{1/e^{t/5} + 2}$$

$$= \frac{4000\cdot 3}{2} = 6000 \text{ caribou}$$