SECTION 4.4: LIMITS OF INDETERMINATE TYPE AND L'HOSPITAL'S RULE

Evaluate:

1. $\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6}$ (type $\frac{0}{0}$) $\frac{4-4}{4-10+6}$

$\boxed{L'H} = \lim_{x\to 2} \frac{2x}{2x-5} = \frac{4}{4-5} = -4$

Algebra $\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x\to 2} \frac{x+2}{x-3} = \frac{4}{2-3} = -4$

2. $\lim_{x\to 0} \frac{\sin x}{x}$ (type $\frac{0}{0}$)

$\boxed{L'H} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1$

3. $\lim_{x\to 0} \frac{\tan(5x)}{\sin(3x)}$ (type $\frac{0}{0}$)

$\boxed{L'H} = \lim_{x\to 0} \frac{(\sec(5x))^2(5)}{(\cos(3x))(3)} = \lim_{x\to 0} \frac{5\cdot\left(\frac{1}{\cos(5x)}\right)^2}{3\cos(3x)} = \frac{5}{3}$ Note $\cos(0)=1$

4. $\cancel{\lim_{u\to\infty} e^{u/10}(u^{-2})}$ $\cancel{\text{(type } \infty\cdot 0 \text{)}}$

$\lim_{u\to\infty} \frac{e^{u/10}}{u^2}$ type $\frac{\infty}{\infty}$

$\boxed{L'H} = \lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{2u}$ type $\frac{\infty}{\infty}$

$\boxed{L'H} = \lim_{u\to\infty} \frac{\left(\frac{1}{10}\right)^2 e^{u/10}}{2}$

$= \infty$

(this says $e^{u/10}$ grows a lot faster than u^2 as $u\to\infty$!)

5. $\lim_{x\to 0} \frac{\cos(4x)}{e^{2x}}$ (type $\frac{1}{1}$)

$= \frac{\cos(0)}{e^0}$

$= 1$

What happens if we blindly try to use L'Hopital's rule?

$\lim_{x\to 0} \frac{-4\sin(4x)}{2e^{2x}}$ is type $\frac{0}{\infty}$

$= 0$

which is NOT the correct limit!

If you use L'H when you're not allowed to it will give you garbage as output.

6. $\lim_{x\to 0} \frac{xe^x}{2^x - 1}$ (type $\frac{0}{0}$)

$\overset{\boxed{L'H}}{=} \lim_{x\to 0} \frac{xe^x + e^x}{2^x \ln(2)}$ "type" $\frac{0+1}{0\ln(2)}$

$= \frac{1}{\ln(2)}$

7. $\lim_{x\to 1^+} \left(\ln(x^4 - 1) - \ln(x^9 - 1)\right)$ (type $\infty - \infty$)

$= \lim_{x\to 1^+} \ln\left(\frac{x^4-1}{x^9-1}\right) = \ln\left(\lim_{x\to 1^+} \frac{x^4-1}{x^9-1}\right)$

$\overset{\boxed{L'H}}{=} \ln\left(\lim_{x\to 1^+} \frac{4x^3}{9x^8}\right) = \ln\left(\frac{4}{9}\right)$

we've got options, here.

$x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$

$x^9 - 1 = (x^3-1)\left((x^3)^2 + x^3(1) + 1\right)$

$= (x-1)(x^2+x+1)(x^6+x^3+1)$

So in fact, we totally can compute this using algebra alone!

8. $\lim_{x\to\infty} \sqrt{x}e^{-x/2}$ (type $\infty \cdot 0$)

$= \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x/2}}$ type $\frac{\infty}{\infty}$

$\overset{\boxed{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2}e^{x/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}\cdot e^{x/2}}$

$= 0$

#9. $\lim_{x\to 0^+} (1+\sin(2x))^{1/x}$ type 1^∞

Let $y = (1+\sin(2x))^{1/x}$. Then

$\ln(y) = \frac{1}{x}\ln(1+\sin(2x))$, so

$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \frac{\ln(1+\sin(2x))}{x}$ type $\frac{0}{0}$

$\overset{\boxed{L'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+\sin(2x)}\cdot\cos(2x)(2)}{1} = \frac{2}{1+0}$.

Therefore $\lim_{x\to 0^+} y = \boxed{\lim_{x\to 0^+} (1+\sin(2x))^{1/x} = e^2}$

since $\lim_{x\to 0^+} \ln(y) = \ln\left(\lim_{x\to 0^+} y\right)$.