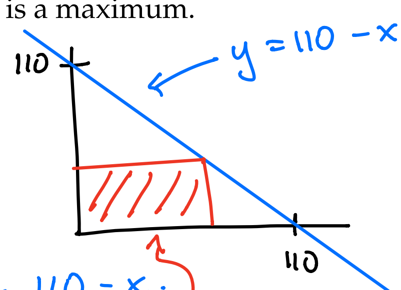
A Framework for Approaching Optimization

1. Read the problem two or three times. Draw pictures. Label them. Pick specific numerical examples, to make the problem concrete. Be creative. Try more than just one approach.
2. Identify the quantity to be minimized or maximized (and which one... min or max).
3. Chose notation and explain what it means.
4. Write the thing you want to maximize or minimize **as a function of one variable**, including a reasonable **domain**.
5. Use calculus to answer the question and justify that your answer is correct.

1. Why does *justification* matter?

Sometimes the answer you think is a local minimum is actually a local maximum or an inflection point – you need to make sure you've found the correct answer to the question!

2. Find two positive numbers whose sum is 110 and whose product is a maximum.



Let x and y be my numbers.

We want $x + y = 110$
and we want to maximize xy.

Let $p = xy$. Since $x + y = 110$, $y = 110 - x$.

Let $p(x) = x(110 - x) = 110x - x^2$.

Observe domain is $[0, 110]$.

Critical points: $p'(x) = 110 - 2x$. (Note $p'(x)$ never undefined)

So $p'(x) = 0 \Rightarrow 110 - 2x = 0 \Rightarrow \frac{110}{2} = 55 = x$.

Is $x = 52$ a max?

$p''(x) = -2 < 0$ so by 2nd deriv test since $p''(55) < 0$

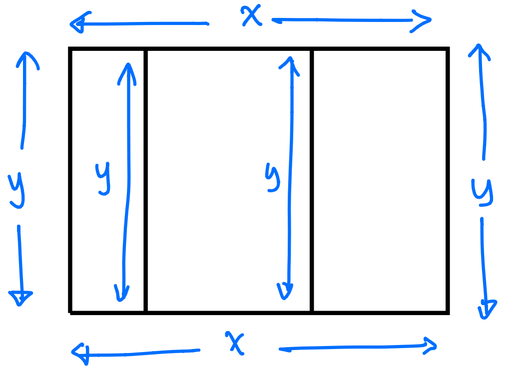
$p(x)$ is $\cap$ at $x = 55$, so yes $\boxed{x = 55 \text{ is a max}}$

Is it a global max? Yes: $p(0) = 0$ and $p(110) = 0$.

$\boxed{\text{ANSWER: } x = 55 \text{ and } y = 55}$

Area is maximized with a square!

3. A rancher has 800 feet of fencing with which to enclose three adjacent rectangular corrals. See figure below. What dimensions should be used so that the enclosed area will be a maximum?



Perimeter $= 2x + 4y = 800$

Want to maximize area $= xy$.

Know $2x + 4y = 800 \Rightarrow$

$4y = 800 - 2x \Rightarrow$

$y = 200 - \frac{1}{2}x$

$A(x) = x\left(200 - \frac{1}{2}x\right)$

$= 200x - \frac{1}{2}x^2$

Domain? If $x = 0$, $y = 200$
If $y = 0$, $x = 400$ } I'm allowing 0 area as a "worst case" scenario.

Domain $= [0, 400]$.

Critical points? $A'(x) = 200 - x$

$A'(x) = 0 \Rightarrow x = 200$. ($A'(x)$ is never undefined)

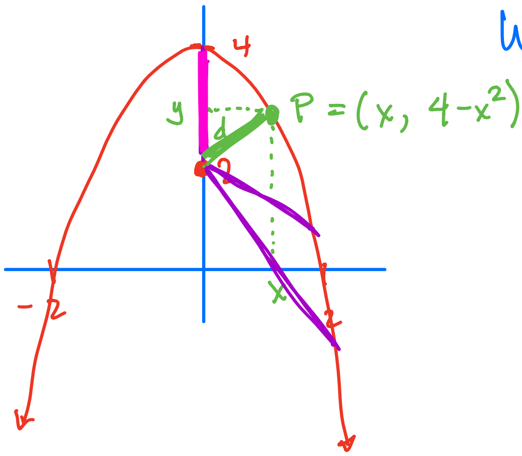
Is it a max? $A''(x) = -1 < 0$ so $A''(200) < 0$.

Yes, $x = 200$ is max. It's a global max also because $A(0) = 0 = A(400)$.

$$\boxed{\text{ANSWER: Dimensions of corral are } 200 \text{ ft} \times 100 \text{ ft}}$$

4. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0,2)$?

- Start by drawing the function $y = 4 - x^2$ and the point $(0,2)$, and identify on your picture what you are trying to minimize.
- Once you have a function — that is, you have made it through part (d) of the Framework — look at the hint at the bottom of the page.)



We want to maximize

$$d(P,(0,2))$$

$$= \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$$

$$= \sqrt{(2-(4-x^2))^2 + (0-x)^2}$$

ugh.

Let's instead maximize $(\text{distance})^2$:

$$d(x) = (2-(4-x^2))^2 + x^2 = (2-4+x^2)^2 + x^2 = (-2+x^2)^2 + x^2$$

$$= 4 - 4x^2 + x^4 + x^2 = x^4 - 3x^2 + 4$$

$$d'(x) = 4x^3 - 6x = 2x(2x^2-3) \Rightarrow d'(x) = 0 \Rightarrow x = 0 \text{ or } x = \sqrt{\tfrac{3}{2}} \text{ or } x = -\sqrt{\tfrac{3}{2}}$$

If we look at the above picture, notice the pink line is longer than the green one. So we expect $x=0$ is NOT a min. Can we justify that?

$d''(x) = 12x^2 - 6$ so $d''(0) < 0$ that is, 0 is a local max!

x		$-\sqrt{1/2}$		$\sqrt{1/2}$	
d''	+	0	−	0	+
d	∪		∩		∪

But $x = \sqrt{\frac{3}{2}}$ & $x = -\sqrt{\frac{3}{2}}$ are both local mins.

$d''(x) = 0 \Rightarrow$
$x^2 = +\frac{6}{12} = \frac{1}{2}$
$d''(-1) = +$, $d''(1) = +$

$4 - \frac{3}{2} = \frac{5}{2}$

$\boxed{\text{ANSWER: the points } \left(\sqrt{\tfrac{3}{2}}, \tfrac{5}{2}\right) \text{ and } \left(-\sqrt{\tfrac{3}{2}}, \tfrac{5}{2}\right) \text{ are closest to } (0,2).}$

HINT: Whenever you are asked to maximize or minimize distance, it is nearly ALWAYS easier to maximize or minimize the square *of the distance. Why?*