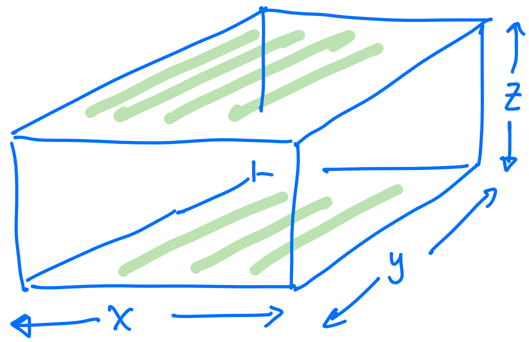
Section 4.7 Applied Optimization (Day 2)

1. A rectangular storage container with lid is to have a volume of 36 cubic inches. The length of the base is three times the width. Material for the base and lid costs \$4 per square inch. Material for the sides costs \$1 per square inch. Find the cost of materials for the least expensive container.



green stuff costs \$4/ sq inch
white stuff costs \$1/sq inch.

Total cost for sides:

$\$1(2xz + 2yz)$

Total cost for top/bottom $= 4(2xy)$.

Know $y = 3x$ and $xyz = 36. \Rightarrow x(3x)\cdot z = 36$

So $z = \frac{36}{3x^2} = \frac{12}{x^2}$.

So $C(x) = 2\overset{x}{x}\left(\overset{z}{\frac{12}{x^2}}\right) + 2\left(\overset{y}{3x}\right)\left(\overset{z}{\frac{12}{x^2}}\right) + 8\overset{x}{x}\left(\overset{y}{3x}\right)$

$= \frac{24}{x} + \frac{72}{x} + 24x^2 = \frac{96}{x} + 24x^2$

Domain? $(0,\infty)$. If x is really big, y is really big & z is really small, but still defined.

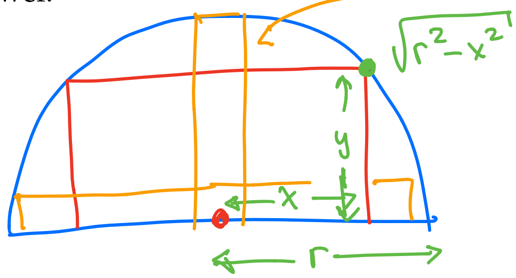
$C'(x) = -\frac{96}{x^2} + 48x$, so $C'(x) = 0 \Rightarrow -\frac{96}{x^2} + 48x = 0 \Rightarrow 96 = 48x^3$

$\Rightarrow \frac{96}{48} = \frac{8\cdot 12}{4\cdot 12} = 2 = x^3$. So $x = \sqrt[3]{2}$. Is this a ~~max~~ min?

$C''(x) = \frac{-96(-2)}{x^3} = \frac{2\cdot 96}{x^3}$ and this is positive when $x > 0$. ∪ so it's a min (which is what we want, to minimize cost). Observe $C(x)$ is CU on $(0,\infty)$ so this must be a global min on the domain.

ANSWER: $x = \sqrt[3]{2}$, $y = 3(\sqrt[3]{2})$, $z = \frac{12}{\sqrt[3]{4}}$ are the dimensions of the box that minimize cost.

2. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.
Hint: The radius r of your circle can be considered a fixed constant. You will expect it to appear in your answer.



Thinking about domain.
If $x \approx 0$, the rectangle is very narrow (and area $\to 0$). If $x \approx r$, rectangle is very flat.

We want to maximize $(2x)(y)$ with the constraint that $y = \sqrt{r^2-x^2}$. Let $A(x) = 2xy = 2x\sqrt{r^2-x^2}$; domain $= [0, r]$.

Then $A'(x) = \cancel{2}x\left(\frac{1}{\cancel{2}}(r^2-x^2)^{-1/2}(-2x)\right) + \sqrt{r^2-x^2}\,(2) = \dfrac{-2x^2}{\sqrt{r^2-x^2}} + 2\sqrt{r^2-x^2}\left(\dfrac{\sqrt{r^2-x^2}}{\sqrt{r^2-x^2}}\right)$

$$= \frac{-2x^2+2(r^2-x^2)}{\sqrt{r^2-x^2}} = \frac{-2x^2+2r^2-2x^2}{\sqrt{r^2-x^2}} = \frac{-4x^2+2r^2}{\sqrt{r^2-x^2}}$$

$A'(x) = 0 \Rightarrow -4x^2+2r^2 = 0 \Rightarrow x^2 = \frac{r^2}{2} \Rightarrow x = \frac{r}{\sqrt{2}}$ or $x = \frac{r}{-\sqrt{2}}$.
(Take only positive because of domain)

$A'(x)$ DNE $\Rightarrow \sqrt{r^2-x^2} = 0 \Rightarrow x = r$ or $x = -r$.

Check which is absolute max using the <u>extreme value theorem</u>:

$A(0) = 0$

$A(r) = 0$

$$A(r/\sqrt{2}) = 2(r/\sqrt{2})\left(\sqrt{r^2-\left(\tfrac{r}{\sqrt{2}}\right)^2}\right)$$

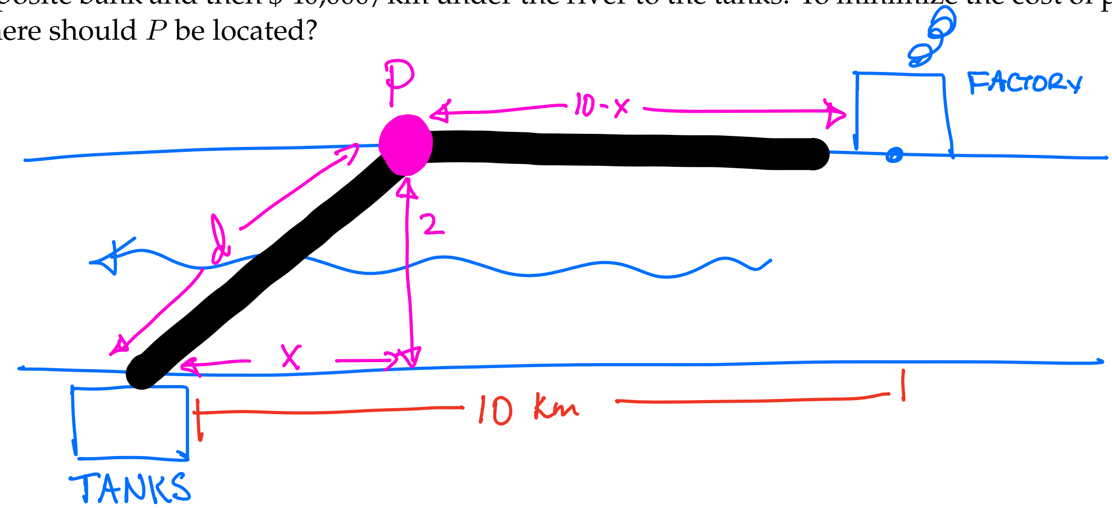
$$= \frac{2r}{\sqrt{2}}\left(\sqrt{r^2-\frac{r^2}{2}}\right) = \frac{2r\cdot r/\sqrt{2}}{\sqrt{2}}$$

$$= \frac{2r^2}{2} = r^2 \leftarrow \text{Abs Max}$$

<u>ANSWER</u>: the area of the largest rectangle is r^2, obtained from a rectangle with dimensions

$$\frac{2r}{\sqrt{2}} \times \frac{r}{\sqrt{2}}$$

3. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 10 km downstream of the refinery. The cost of laying pipe is \$ 10,000/ km over land to a point P on the opposite bank and then \$ 40,000/km under the river to the tanks. To minimize the cost of pipeline, where should P be located?



Know $2^2 + (x)^2 = d^2 \Rightarrow d = \sqrt{4 + x^2}$

Know total cost $= 4d + 1(10-x)$ (in tens of thousands of \$)

So $C(x) = 4\sqrt{4+x^2} + 10 - x$

Domain: $[0, 10]$
(x varies from next to the factory to directly opposite the tanks)

Critical points:

$C'(x) = \cancel{4}^{\,2}\left(\frac{1}{\cancel{2}}(4+x^2)^{-1/2}(2x)\right) - 1$

$= \dfrac{4x}{\sqrt{4+x^2}} - 1$

Observe $\sqrt{4+x^2} > 0$ so only critical points are where $C'(x) = 0 \Rightarrow$

$\dfrac{4x}{\sqrt{4+x^2}} = 1 \Rightarrow 4x = \sqrt{4+x^2} \Rightarrow 16x^2 = 4 + x^2 \Rightarrow 15x^2 = 4$

$\Rightarrow x = \dfrac{2}{\sqrt{15}}$ or $\dfrac{-2}{\sqrt{15}}$ (not in domain).

$C'\left(\frac{1}{100}\right) = \dfrac{4/100}{\sqrt{4 + \frac{1}{10000}}} - 1 \approx -1 + \dfrac{4}{200} < 0$

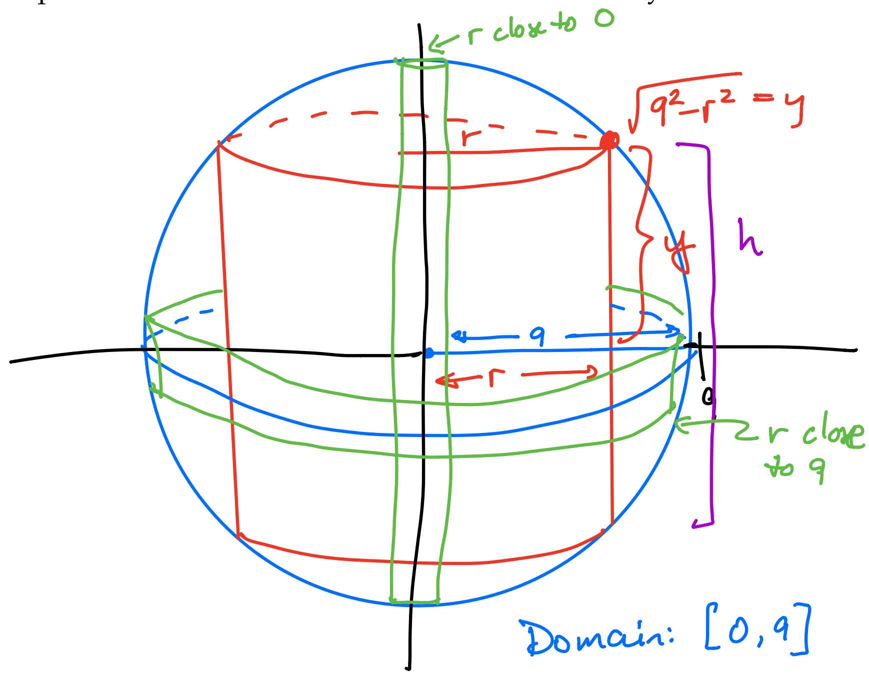
x		$\frac{2}{\sqrt{15}}$	
test	$1/100$		1
C'	−	0	+
C	↘	$\boxed{\text{MIN}}$	↗

$C'(1) = \dfrac{4}{\sqrt{5}} - 1 = \dfrac{4 - \sqrt{5}}{\sqrt{5}}$ and $\sqrt{5} < 4$ so $C'(1) > 0$

$\boxed{\text{ANSWER}}$

So $x = \frac{2}{\sqrt{15}}$ is a local & abs min and P should be located $\frac{2}{\sqrt{15}}$ miles upstream of the storage tanks.

4. A right circular cylinder is inscribed in a sphere of radius 9. Find the dimensions of the cylinder with largest possible volume.



Maximize cylinder volume

$= \pi r^2 h$

$\Rightarrow V = \pi r^2 (2y)$

Subject to the constraint that

$y = \sqrt{9^2 - r^2}$

So $V(r) = 2\pi r^2 \sqrt{9^2 - r^2}$

Domain: $[0, 9]$

Critical points:

$V'(r) = \cancel{2}\pi r^2 \left(\frac{1}{\cancel{2}}(81-r^2)^{-1/2}(-2r)\right) + \sqrt{81-r^2}\,(4\pi r)$ ← This is the hard way. It's easier to write $V(r) = 2\pi\sqrt{81r^4 - r^6}$.

$$= \frac{-2r^3\pi}{\sqrt{81-r^2}} + \sqrt{81-r^2}\,(4\pi r) = \frac{-2r^3\pi + 4\pi r(81-r^2)}{\sqrt{81-r^2}}$$

$$= \frac{-2\pi r^3 - 4\pi r^3 + 4\cdot 81\pi r}{\sqrt{81-r^2}} = \frac{-6\pi r^3 + 4\cdot 81\pi r}{\sqrt{81-r^2}} = \frac{-6\pi r(r^2 - 54)}{\sqrt{81-r^2}}$$

Critical points where $V'(r)$ undefined, at $r = 9$, and where $V'(r) = 0 \Rightarrow$

$r = 0$ or $r^2 = 54 \Rightarrow r = \sqrt{9\cdot 6} = 3\sqrt{6}$ or $r = -3\sqrt{6}$ (out of domain).

Observe that on $[0, 9]$, by the extreme value theorem, $V(r)$ must have an absolute max & min, and $V(0) = 0$, $V(9) = 0$, so $r = 3\sqrt{6}$ must be where the absolute max is! But we can verify:

$V(3\sqrt{6}) = 2\pi(54)\sqrt{81-54} = 2\pi(54)\sqrt{27}$ is bigger than 0

So must be the absolute max.

$$\boxed{\text{ANSWER: DIMENSIONS are radius} = 3\sqrt{6} \text{ and height} = 2\sqrt{81-54} = 2\sqrt{27} = 6\sqrt{3}.}$$