SECTION 4.8 NEWTON'S METHOD

Newton's Method is an iterative rule for finding roots.

Given: $F(x)$
Want: a so that $F(a) = 0$
Guess: x_0 close to a

Plug in and Repeat:
`newX = oldX - F(oldX)/F'(oldX)`
In math language:

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}$$

1. Let $F(x) = x^2 - 2$.

   (a) Using elementary algebra, find a such that $F(a) = 0$. (Find a exactly and find a decimal approximation with at least 9 decimal places.)

   $x^2 - 2 = 0 \Rightarrow x = \sqrt{2}$ or $x = -\sqrt{2}$

   $\sqrt{2} \approx 1.414213562$

   (b) Find a formula for x_{k+1}. Simplify it.

   $$x_{k+1} = x_k - \left(\frac{(x_k)^2 - 2}{2x_k}\right) = \frac{2x_k^2 - x_k^2 + 2}{2x_k} = \frac{x_k^2 + 2}{2x_k} = \frac{x_k}{2} + \frac{1}{x_k}$$

   (c) Using an initial guess of $x_0 = 2$, complete 4 iterations of Newton's method to find x_4 and compare your answer to the one in part (a). ← Work in fractions or using at least 10 digits to avoid roundoff error.

   $x_0 = 2$

   $x_1 = \frac{2}{2} + \frac{1}{2} = \frac{3}{2} = 1.5$

   $x_2 = \frac{3/2}{2} + \frac{1}{3/2} = \frac{3}{4} + \frac{2}{3} = \frac{9+8}{12} = \frac{17}{12} = \underline{1.41}666\ldots$

   $x_3 = \frac{17/12}{2} + \frac{1}{17/12} = \frac{17}{24} + \frac{12}{17} = \frac{577}{408} = \underline{1.41421}5686$ 5 digits the same!

   $x_4 = \frac{577/408}{2} + \frac{1}{577/408} = 1.414213562$ ← all 9 digits the same!

2. This page is intended to illustrate *how* Newton's Method works.

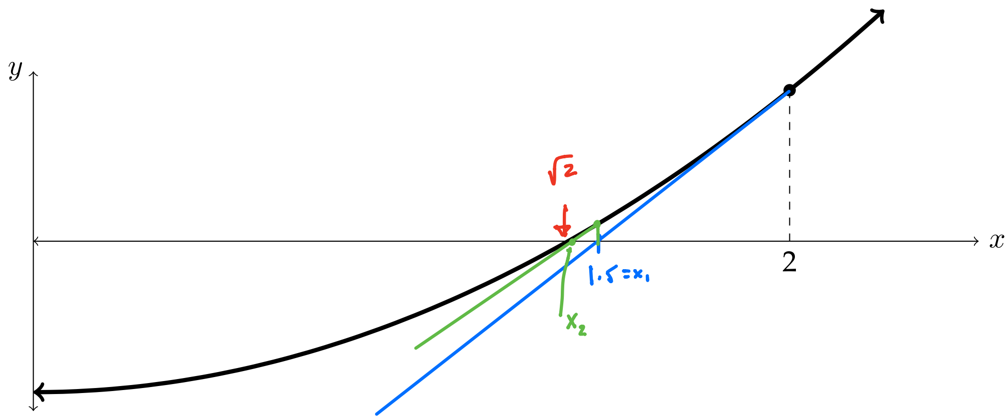
Again, consider the function $F(x) = x^2 - 2$.

(a) Find the linearization $L(x)$ of $F(x)$ at $x = 2$. Leave your answer in point-slope form.

$F'(2) = 4$, $F(2) = 2$

$L(x) = 4(x-2) + 2$

(b) I've graphed $F(x)$ for you below. Mark where $\sqrt{2}$ is on this diagram and add to this diagram the graph of $L(x)$. Use a ruler.



(c) Find the number x_1 such that $L(x_1) = 0$.

Solve $4(x-2)+2=0 \Rightarrow 4(x-2) = -2 \Rightarrow x-2 = -\frac{1}{2} \Rightarrow x = \frac{3}{2}$

So $x_1 = \frac{3}{2} = 1.5$

(d) In the diagram above, label the point x_1 on the x-axis.

(e) Let's do it again! Find the linearization $L(x)$ of $F(x)$ at $x = x_1$.

$F'(3/2) = 2(3/2) = 3$ and $F(3/2) = (3/2)^2 - 2 = \frac{9}{4} - \frac{8}{4} = 1/4$

$L(x) = 3(x - 3/2) + 1/4$

(f) Add the graph of this new linearization to your diagram above.

(g) Find the number x_2 such that $L(x_2) = 0$. Then label the point $x = x_2$ in the diagram.

Solve $3(x - 3/2) + 1/4 = 0 \Rightarrow 3(x - 3/2) = -1/4 \Rightarrow x - 3/2 = \frac{-1}{12}$

$\Rightarrow x = \frac{-1}{12} + \frac{3}{2} = \frac{-1}{12} + \frac{18}{12} = \frac{17}{12}$

(h) Compare your numbers for x_1 and x_2 to those on the previous page. They should be the same. They are the same! Hooray!

(i) Let's be a little more systematic. Suppose we have an estimate x_k for $\sqrt{2}$.

- Compute $F(x_k)$. $F(x_k) = (x_k)^2 - 2$
- Compute $F'(x_k)$. $F'(x_k) = 2x_k$
- Compute the linearization of $F(x)$ at $x = x_k$.

$$L(x) = 2x_k(x - x_k) + (x_k)^2 - 2$$

- Find the number x_{k+1} such that $L(x_{k+1}) = 0$. You should try to find as simple an expression as you can. Compare this to the formula we used on problem 1b from page 1.

Solve $2x_k(x - x_k) + (x_k)^2 - 2 = 0 \Rightarrow 2x_k(x - x_k) = 2 - x_k^2$

$\Rightarrow x = x_k + \dfrac{2 - (x_k)^2}{2x_k} = x_k + \dfrac{1}{x_k} - \dfrac{1}{2}(x_k) = \dfrac{x_k}{2} + \dfrac{1}{x_k}$

the same formula!

3. Try to solve

$$e^{-x} - x = 0$$

by hand.

$\dfrac{1}{e^x} = x \Rightarrow e^x = \dfrac{1}{x} \Rightarrow x = \ln\left(\dfrac{1}{x}\right)$:(

$xe^x = 1 \Rightarrow$:(We don't know how to solve this.

4. Explain why there is a solution between $x = 0$ and $x = 1$.

$f(x) = e^{-x} - x \Rightarrow f(0) = \dfrac{1}{e^0} - 0 = 1$

$f(1) = \dfrac{1}{e} - 1 < \dfrac{1}{2} - 1 = -\dfrac{1}{2} < 0$

Note $2 < e < 3$ so $\frac{1}{2} > \frac{1}{e} > \frac{1}{3}$

So by IVT, there is a solution to $f(x) = 0$ in $(0, 1)$.

5. Starting with $x_0 = 1$, find an approximation of the solution of $e^{-x} - x = 0$ to 6 decimal places. During your computation, keep track of each x_k to at least 9 decimal places of accuracy.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{e^{-x_k} - x_k}{-e^{-x_k} + 1} = x_k - \left(\frac{1/e^{x_k} - x_k}{-1/e^{x_k} + 1}\right)\frac{e^{x_k}}{e^{x_k}} = x_k + \frac{1 - x_k e^{x_k}}{e^{x_k} - 1}$$

$x_0 = 1$

$x_1 = 1 + \dfrac{1 - e}{e - 1} = 0.537882842739990$

$x_2 = 0.566986991405 41$

$x_3 = 0.567143285 98912$

$x_4 = 0.567143290409 78$ ← 7 decimals agreement w/ x_3

$x_5 = 0.56714329040978$ ← lots of decimals agreement w/ x_4