

Definite Integrals and Areas "under" Curves

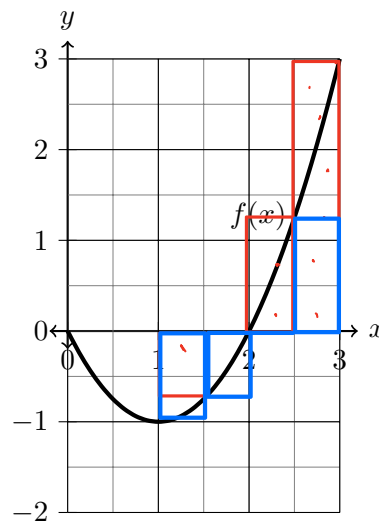
1. Estimate the area (actually draw rectangles) under $f(x) = x^2 - 2x$ on $[1, 3]$ with $n = 4$ using the

(a) Right-hand endpoints Width = $\frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

$$\begin{aligned} \text{area} &= \frac{1}{2} f\left(\frac{3}{2}\right) + \frac{1}{2} f\left(\frac{4}{2}\right) + \frac{1}{2} f\left(\frac{5}{2}\right) + \frac{1}{2} f\left(\frac{6}{2}\right) \\ &= \frac{1}{2} \left[\left(\frac{9}{4} - \frac{12}{4}\right) + \left(\frac{16}{4} - \frac{16}{4}\right) + \left(\frac{25}{4} - \frac{20}{4}\right) + \left(\frac{36}{4} - \frac{24}{4}\right) \right] = \frac{1}{2} \left[-\frac{3}{4} + 0 + \frac{5}{4} + \frac{12}{4} \right] = \frac{1}{2} \left[\frac{14}{4} \right] = \frac{7}{4} \end{aligned}$$

(b) Left-hand endpoints

$$\begin{aligned} \text{Area} &= \frac{1}{2} (f(1)) + \frac{1}{2} f\left(\frac{3}{2}\right) + \frac{1}{2} f\left(\frac{4}{2}\right) + \frac{1}{2} f\left(\frac{5}{2}\right) \\ &= \frac{1}{2} \left[(1-2) + \left(-\frac{3}{4}\right) + 0 + \frac{5}{4} \right] = \frac{1}{2} \left[-\frac{4}{4} - \frac{3}{4} + \frac{5}{4} \right] = -\frac{1}{4} \end{aligned}$$



Definition of a Definite Integral If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 = (a), x_1, x_2, \dots, x_n = (b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be **sample points**¹ in these subintervals, so x_i^* lies in the i -th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x,$$

provided this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

¹ For example, we could choose our sample points to be right-hand endpoints, left-hand endpoints, midpoints, a combination of these, or any other sample points in the interval that we choose!

2. Consider again $f(x) = x^2 - 2x$ on the interval $[1, 3]$. Suppose that we are dividing the interval $[1, 3]$ into n subintervals. (Think about your answers to #1.)

- (a) What is the length of each subinterval? $\frac{3-1}{n} = \frac{2}{n}$
- (b) What is the right-hand endpoint of the first subinterval? $1 + \frac{2}{n}$
 What is the height of the first right-hand rectangle? $f(1 + \frac{2}{n})$
- (c) What is the right-hand endpoint of the second subinterval? $1 + 2(\frac{2}{n})$
 What is the height of the second right-hand rectangle? $f(1 + 2(\frac{2}{n}))$
- (d) What is the right-hand endpoint of the third subinterval? $1 + 3(\frac{2}{n})$
 What is the height of the third right-hand rectangle? $f(1 + 3(\frac{2}{n}))$
- (e) What is the right-hand endpoint of the i^{th} rectangle? $1 + i(\frac{2}{n})$
 What is the height of the i -th right-hand rectangle? $f(1 + i(\frac{2}{n}))$
 What is the **area** of the i -th right-hand rectangle? $(\frac{2}{n}) f(1 + \frac{2i}{n})$

3. Using your answers to the previous problem, write down a limit that equals $\int_1^3 x^2 - 2x \, dx$.

$$\int_1^3 x^2 - 2x \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(1 + \frac{2i}{n}\right)^2 - 2\left(1 + \frac{2i}{n}\right) \right] \left(\frac{2}{n}\right)$$

4. Write down a limit that equals $\int_2^8 e^x \, dx$, using right-hand endpoints as your sample points.

$$\text{Width} = \Delta x = \frac{8-2}{n} = \frac{6}{n}$$

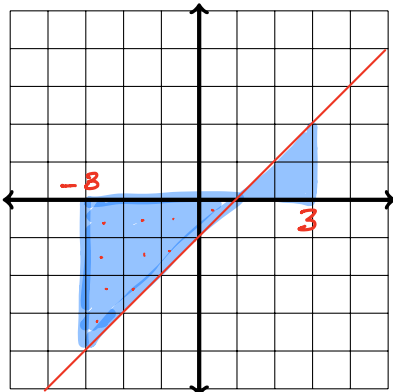
$$x_i = 2 + i\Delta x = 2 + \frac{6i}{n}$$

$$\int_2^8 e^x \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(e^{2 + \frac{6i}{n}} \right) \left(\frac{6}{n} \right)$$

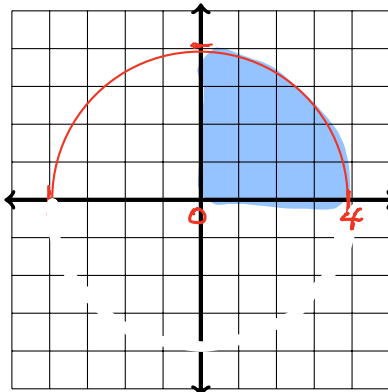
A definite integral represents the **signed** area under a curve (that is, the signed area between the curve and the x-axis). If a curve is above the x-axis that area is positive; if the curve is below the x-axis the area is negative.

5. Evaluate the following definite integrals by drawing the function and interpreting the integral in terms of areas. Shade in the area you are computing with the integral and then actually compute the integral by adding up areas.

(a) $\int_{-3}^3 (x-1) \, dx = -8 + 2 = -6$

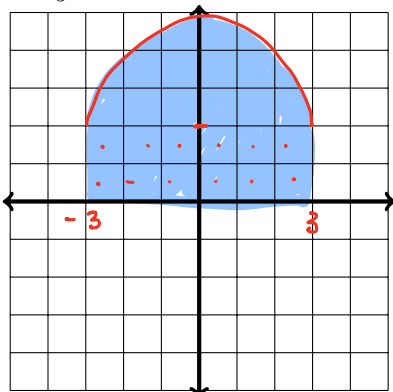


(b) $\int_0^4 \sqrt{16-x^2} \, dx = 4\pi$

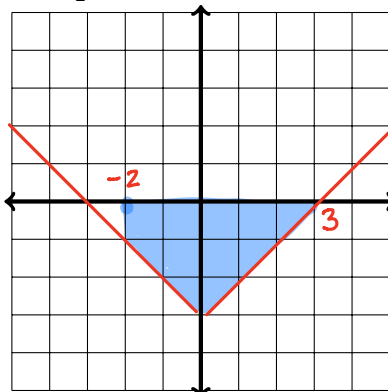


Total circle = $\pi(4)^2$
so 1/4 circle is 4π

(c) $\int_{-3}^3 (2 + \sqrt{9-x^2}) \, dx = 12 + \frac{9\pi}{2}$



$\int_{-2}^3 (|x-3|) \, dx = 8.5$

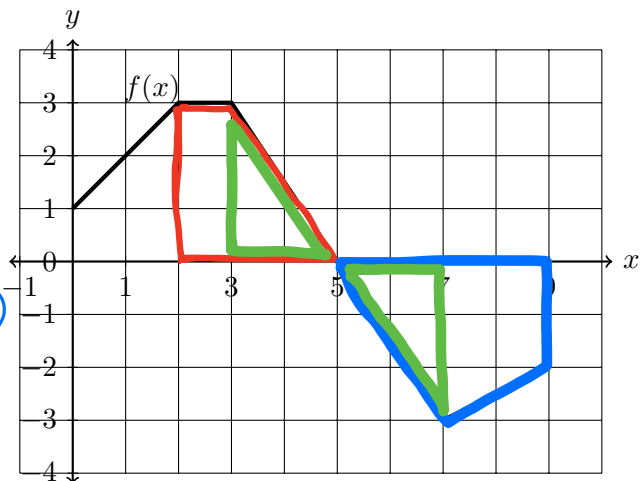


6. The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

(a) $\int_2^5 f(x) dx = 3 + \frac{1}{2}(2)(3) = 6$

(b) $\int_5^9 f(x) dx = -(4 + \frac{1}{2}(2)(3) + \frac{1}{2}(2)(1))$
 $= -(4 + 3 + 1) = -8$

(c) $\int_3^7 f(x) dx = 0$



Properties of the Definite Integral:

- $\int_a^a f(x) dx = 0$
- $\int_a^b c dx = c(b-a)$
- $\int_a^b cf(x) dx = c \int_a^b f(x) dx$
- $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
- $\int_b^a f(x) dx = - \int_a^b f(x) dx$

7. Using the fact that $\int_0^1 x^2 dx = \frac{1}{3}$ and $\int_1^2 x^2 dx = \frac{7}{3}$, evaluate the following using the properties of integrals.

(a) $\int_1^0 x^2 dx$	(b) $\int_0^1 5x^2 dx$	(c) $\int_0^1 (4 + 3x^2) dx$	(d) $\int_0^2 x^2 dx$
$= - \int_0^1 x^2 dx$	$= 5 \int_0^1 x^2 dx$	$= \int_0^1 4 dx + 3 \int_0^1 x^2 dx$	$= \int_0^1 x^2 dx + \int_1^2 x^2 dx$
$= -\frac{1}{3}$	$= 5 \left(\frac{1}{3}\right)$	$= 4(1) + 3\left(\frac{1}{3}\right)$	$= \frac{1}{3} + \frac{7}{3}$
	$= \frac{5}{3}$	$= 7$	$= \frac{8}{3}$