SECTION 5-3: THE FUNDAMENTAL THEOREM OF CALCULUS, PART 2

The Fundamental Theorem of Calculus (Part 2) If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is **any antiderivative** of f, that is, is a function such that $F' = f$. To evaluate, we write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

1. Evaluate the following integrals.

(a) $\displaystyle\int_0^1 x^2\,dx$

$$\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$$

(b) $\displaystyle\int_1^4 (1 + 3y - y^2)\,dy = y + \frac{3y^2}{2} - \frac{y^3}{3}\Big|_1^4$

$$= \left(4 + \frac{3(4)^2}{2} - \frac{4^3}{3}\right) - \left(1 + \frac{3}{2} - \frac{1}{3}\right)$$

$$= 4 + 24 - \frac{64}{3} - 1 + \frac{3}{2} + \frac{1}{3} = 27 - 21 + 3/2 = 15/2$$

2. Review from §4.9: To compute integrals effectively you **must** have your basic antidifferentiation formulas down. You should know that antiderivatives to the following functions. We are using the $\int$ symbol to mean "find the antiderivative" of the function right after the symbol.

Antiderivatives of common functions:

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
- $\int \sin x\,dx = -\cos(x) + c$
- $\int \cos x\,dx = \sin(x) + c$
- $\int \sec^2 x\,dx = \tan(x) + c$
- $\int \sec x \tan x\,dx = \sec(x) + c$
- $\int \csc^2 x\,dx = -\cot(x) + c$
- $\int \csc x \cot x\,dx = -\csc(x) + c$
- $\int e^x\,dx = e^x + c$
- $\int a^x\,dx = \frac{a^x}{\ln(a)} + c$
- $\int \frac{1}{1+x^2}\,dx = \arctan(x) + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + c$
- $\int \frac{1}{x}\,dx = \ln|x| + c$

$a^x = e^{\ln(a^x)} = e^{x\ln(a)}$

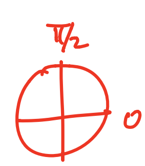
so $\frac{d}{dx}(a^x) = a^x \ln(a)$

3. Evaluate the following integrals.

(a) $\displaystyle\int_2^5 \frac{3}{x}\,dx = 3\int_2^5 \frac{1}{x}\,dx$

$$= 3\ln|x|\Big|_2^5 = 3\ln(5) - 3\ln(3)$$

(b) $\displaystyle\int_0^{\pi/2} \cos x\,dx$

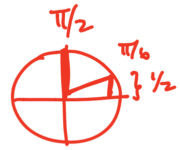


$$= \sin(x)\Big|_0^{\pi/2}$$

$$= \sin(\pi/2) - \sin(0)$$

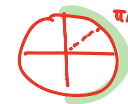
$$= 1$$

4. Evaluate the following integrals.



$\arctan 1 = \theta \Rightarrow$
$\tan\theta = 1$

$\tan x = \frac{\sin x}{\cos(x)}$



(a) $\displaystyle\int_1^8 \sqrt[3]{x}\,dx$

$= \int_1^8 x^{1/3}\,dx$

$= \frac{3x^{4/3}}{4}\Big|_1^8 = \frac{3\cdot(8^{1/3})^4}{4} - \frac{3(1)^4}{4}$

$= \frac{3\cdot 16}{4} - \frac{3}{4} = 12 - 3/4 = \frac{45}{4}$

(b) $\displaystyle\int_{\pi/6}^{\pi/2} \csc x \cot x\,dx$

$= -\csc(x)\Big|_{\pi/6}^{\pi/2}$

$= \frac{-1}{\sin(x)}\Big|_{\pi/6}^{\pi/2}$

$= -1-(-2)$

$= 1$

(c) $\displaystyle\int_0^1 \frac{9}{1+x^2}\,dx$

$= 9\arctan(x)\Big|_0^1$

$= 9(\arctan(1) - \arctan(0))$

$= 9\left(\frac{\pi}{4}\right) - 9(0) = \frac{9\pi}{4}$

5. We do not have any product or quotient rules for antidifferentiation. To evaluate an integral that is expressed as a product or quotient you must try to manipulate the integrand (the stuff inside the $\int$ sign) to look like something you know how to antidifferentiate. The following integrals are examples of this. Evaluate the following integrals.

(a) $\displaystyle\int_1^3 \frac{x^3+3x^6}{x^4}\,dx = \int_1^3 \frac{1}{x} + 3x^2\,dx$

$= \ln|x| + \frac{\cancel{3}\cdot x^3}{\cancel{3}}\Big|_1^3$

$= \ln(3) + 27 - \ln(1) - 1$

$= \ln(3) + 26$

(b) $\displaystyle\int_0^1 x(3+\sqrt{x})\,dx = \int_0^1 3x + x^{3/2}\,dx$

$= \left(\frac{3x^2}{2} + \frac{2x^{5/2}}{5}\right)\Big|_0^1$

$= \frac{3}{2} + \frac{2}{5} - 0$

$= \frac{15+4}{10} = \frac{19}{10}$

6. Evaluate the following integrals.

(a) $\displaystyle\int_0^2 (5^x + x^5)\,dx$

$= \frac{5^x}{\ln(5)} + \frac{x^6}{6}\Big|_0^2$

$= \left(\frac{25}{\ln(5)} + \frac{32}{6}\right) - \frac{1}{\ln(5)}$

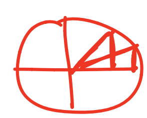
$= 24/\ln(5) + 16/3$

(b) $\displaystyle\int_{1/2}^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}\,dx$

$= \arcsin(x)\Big|_{1/2}^{\sqrt{2}/2} = \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin(1/2)$

$= \pi/4 - \frac{\pi}{6}$

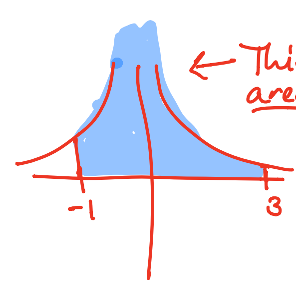
$= \frac{3\pi - 2\pi}{12} = \pi/12$



7. What is wrong with the following calculation? (Hint: draw a picture!)

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Big|_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

FTC 2 does not apply, because $f(x) = 1/x^2$ is not continuous on $[-1, 3]$.



← This is the area indicated by the integral!