Section 5-5: Substitution (Day 1)

1. Compute $\int t\sin(t^2+1)\,dt$

Let $u = t^2+1$. Then $\frac{du}{dt} = 2t \Rightarrow \frac{du}{2t} = dt$

So $\int t \sin(t^2+1)\,dt = \int \cancel{t} \sin(u) \cdot \frac{du}{2\cancel{t}} = \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(u) + C$

$= \boxed{-\frac{1}{2}\cos(t^2+1) + C}$

2. Compute $\int e^{4x-9}\,dx$

Let $u = 4x-9$. Then $\frac{du}{dx} = 4 \Rightarrow \frac{du}{4} = dx$

So $\int e^{4x-9}\,dx = \int e^u \cdot \frac{du}{4} = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C = \boxed{\frac{1}{4}e^{4x-9} + C}$

3. Compute $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Let $u = \sqrt{x}$. Then $\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2\sqrt{x}\,du = dx$.

So $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{\cancel{\sqrt{x}}}(2\cancel{\sqrt{x}}\,du) = 2\int e^u\,du = 2e^u + C$

$= \boxed{2e^{\sqrt{x}} + C}$

4. Compute $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}}\Big|_1^4 = 2e^{\sqrt{4}} - 2e^{\sqrt{1}}$

$= 2e^2 - 2e$

$= \boxed{2e(e-1).}$

If we had not already computed the antiderivative, we could have changed the bounds!

Let $u = \sqrt{x}$. Then when $x=1$, $u = \sqrt{1} = 1$ and when $x=4$, $u=\sqrt{4} = 2$. So

$\int_{x=1}^{x=4} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_{u=1}^{u=2} \frac{e^u}{\cancel{\sqrt{x}}}(2\cancel{\sqrt{x}}\,du)$

$= \int_{u=1}^{u=2} 2e^u\,du = 2e^u\Big|_1^2 = 2e^2 - 2e^1.$

5. Compute $\displaystyle\int \frac{\arctan(x)}{1+x^2}\,dx$

Let $u = \arctan(x)$. Then $\frac{du}{dx} = \frac{1}{1+x^2}$ so $du\,(1+x^2) = dx$.

So $\displaystyle\int \frac{\arctan(x)}{1+x^2}\,dx = \int \frac{u}{1+x^2}(1+x^2)\,du = \int u\,du = \frac{u^2}{2} + C$

$= \boxed{\frac{(\arctan(x))^2}{2} + C}$

6. Compute $\displaystyle\int \frac{x^3}{\sqrt{1-x^4}}\,dx$

Let $u = 1-x^4$. Then $\frac{du}{dx} = -4x^3 \Rightarrow \frac{du}{-4x^3} = dx$.

So $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \int \frac{x^3}{\sqrt{u}}\left(\frac{du}{-4x^3}\right) = -\frac{1}{4}\int u^{-1/2}\,du = -\frac{1}{4}\,\frac{u^{1/2}}{1/2} + C = -\frac{1}{2}\,u^{1/2} + C$

$= -\frac{1}{2}\sqrt{1+x^4} + C.$

7. Compute $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx.$ $\displaystyle = \int \frac{x}{\sqrt{1-(x^2)^2}}\,dx.$

Let $u = x^2$. Then $\frac{du}{dx} = 2x \Rightarrow \frac{du}{2x} = du$. So

$\displaystyle\int \frac{x}{\sqrt{1-(x^2)^2}}\,dx = \int \frac{x}{\sqrt{1-u^2}}\left(\frac{du}{2x}\right) = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin(x^2) + C$

8. Compute $\displaystyle\int_0^{\pi/6} \frac{\sin(t)}{(cos(t))^2}\,dt$ two ways: (1) by computing the antiderivative using substitution and then using FTC2 to evaluate using the original bounds; (2) by substituting and changing the bounds to match the substitution.

① Let $u = \cos(t)$. Then $\frac{du}{dt} = -\sin(t) \Rightarrow$ $\frac{-du}{\sin(t)} = dt$. So $\displaystyle\int_{t=0}^{t=\pi/6} \frac{\sin(t)}{(\cos(t))^2}\,dt =$

$\displaystyle\int_{t=0}^{t=\pi/6} \frac{\cancel{\sin(t)}}{u^2}\left(\frac{-du}{\cancel{\sin(t)}}\right) = \int_{t=0}^{t=\pi/6} -u^{-2}\,du = -\frac{u^{-1}}{-1}\Bigg|_{t=0}^{t=\pi/6}$

$= \frac{1}{\cos(t)}\Bigg|_0^{\pi/6} = \frac{2}{\sqrt{3}} - 1$

② $u = \cos(t) \Rightarrow \frac{du}{dt} = -\sin(t)$

$\Rightarrow \frac{-du}{\sin(t)} = dt$. And if $t=0$, $u=\cos(0)=1$

and if $t=\pi/6$, $u=\cos(\pi/6) = \frac{\sqrt{3}}{2}$. So

$\displaystyle\int_{t=0}^{t=\pi/6} \frac{\sin(t)}{(\cos(t))^2}\,dt = \int_{u=1}^{u=\sqrt{3}/2} \frac{\sin(t)}{u^2}\left(\frac{-du}{\sin t}\right) =$

$\displaystyle\int_1^{\sqrt{3}/2} -u^{-2}\,du = \frac{1}{u}\Bigg|_1^{\sqrt{3}/2} = \frac{2}{\sqrt{3}} - 1.$