LECTURE NOTES: 4-5 CURVE SKETCHING (PART 1)

$\boxed{\text{GUIDELINES OF ALL CURVE SKETCHING PROBLEMS}}$ For each item below, write out in your own words how you actually find that item.

A. **Domain.** Find the domain of the function.

Look for "allowable" x-values avoiding ① zero in denominator, ② negative #'s under square root, ③ 0 or neg #'s in natural log, etc.

B. **Intercepts** Find any x- or y-intercepts.

x-intercept : set $y=0$. Solve for x

y-intercept : set $x=0$. Solve for y.

C. **Symmetry** Determine if the function is even or odd.

- use even powers or odd powers
- $\sin x$ is odd, $\cos x$ is even
- Some functions are neither

D. **Asymptotes** Identify any vertical or horizontal asymptotes.

$x=a$ is a vertical asymptote if $\lim_{x\to a^{\pm}} f(x) = \pm\infty$

$y=b$ is a horizontal asymptote if $\lim_{x\to\pm\infty} f(x) = b$

E. **Intervals of Increase or Decrease** Determine the intervals where the function is increasing and where the function is decreasing.

$f' > 0$ on I, then f is increasing on I

$f' < 0$ on I, then f is decreasing on I

F. **Local Maximum and Minimum Values** Identify any local maximums and minimums and where they occur.

If $f'(c) = 0$ or $f'(c)$ is undefined and c is in the domain of $f(x)$, then $f(c)$ local max $\frown$ if f' is pos→neg; $f(c)$ local min $\smile$ if f' is neg→pos.

G. **Concavity and Points of Inflection** Find the intervals where the function is concave up and where the function is concave down. Identify any inflection points.

- $f'' > 0 \Rightarrow$ ccup $\cup$
- $f'' < 0 \Rightarrow$ ccdown $\cap$
- inflection point, (x,y), where concavity changes

H. **Sketch the Curve** Plot the curve. Include and label all the bits and pieces above.

- Include important points.

$\boxed{\text{PRACTICE PROBLEM}}$ Sketch the curve $y = \dfrac{2x^2}{x^2-4}$ $= \dfrac{2x^2}{(x+2)(x-2)}$

(a) Find the domain.

all real numbers except $x = \pm 2$. OR $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$

(b) Find the x and y-intercepts.

If $x=0$, then $y = \frac{0}{-4} = 0$. If $y=0$, then $0 = \frac{2x^2}{x^2-4}$. So $x=0$.

Ans: x-intercept is 0, y-intercept is 0.

(c) Find the symmetries of the curve.

all powers are even.

answer: $f(x)$ is even

(d) Determine the asymptotes.

- Find the horizontal asymptotes.

$\lim\limits_{x\to\infty} \frac{2x^2}{x^2-4} = 2$. $\lim\limits_{x\to-\infty} \frac{2x^2}{x^2-4} = 2$. ANS: $y=2$

- Find the vertical asymptotes.

$\lim\limits_{x\to-2^+} \frac{2x^2}{x^2-4} = -\infty$; $\lim\limits_{x\to 2^+} \frac{2x^2}{x^2-4} = +\infty$ ANS $x=2$, $x=-2$

(e) Determine where the function is increasing/ decreasing.

$$y = \frac{2x^2}{x^2-4}$$

$$y' = \frac{(x^2-4)(4x) - 2x^2(2x)}{(x^2-4)^2}$$

$$= \frac{4x^3 - 16x - 4x^3}{(x^2-4)^2}$$

$$= -\frac{16x}{(x^2-4)^2}$$

critical pts: $x=0$

0

+++ 0 - - -

sign of y'

ANSWER

f increases on $(-\infty,-2) \cup (-2,0)$ and

decreases on $(0,2) \cup (2,\infty)$

(f) Find the local maximum/ minimum values.

local max at $x=0$

max value is $f(0)=0$.

(g) Find the intervals of concavity/inflection points.

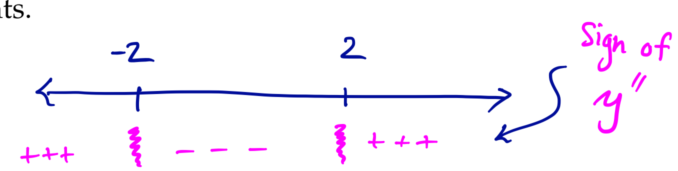
$$y' = \frac{-16x}{(x^2-4)^2}$$

$$y'' = \frac{(x^2-4)^2(-16) + 16x \cdot 2(x^2-4)^1(2x)}{(x^2-4)^4}$$

$$= \frac{16(x^2-4)\left[-(x^2-4) + 4x^2\right]}{(x^2-4)^4}$$

$$= \frac{16\left[3x^2+4\right]}{(x^2-4)^3}$$

← numerator never zero! always positive



ANS:

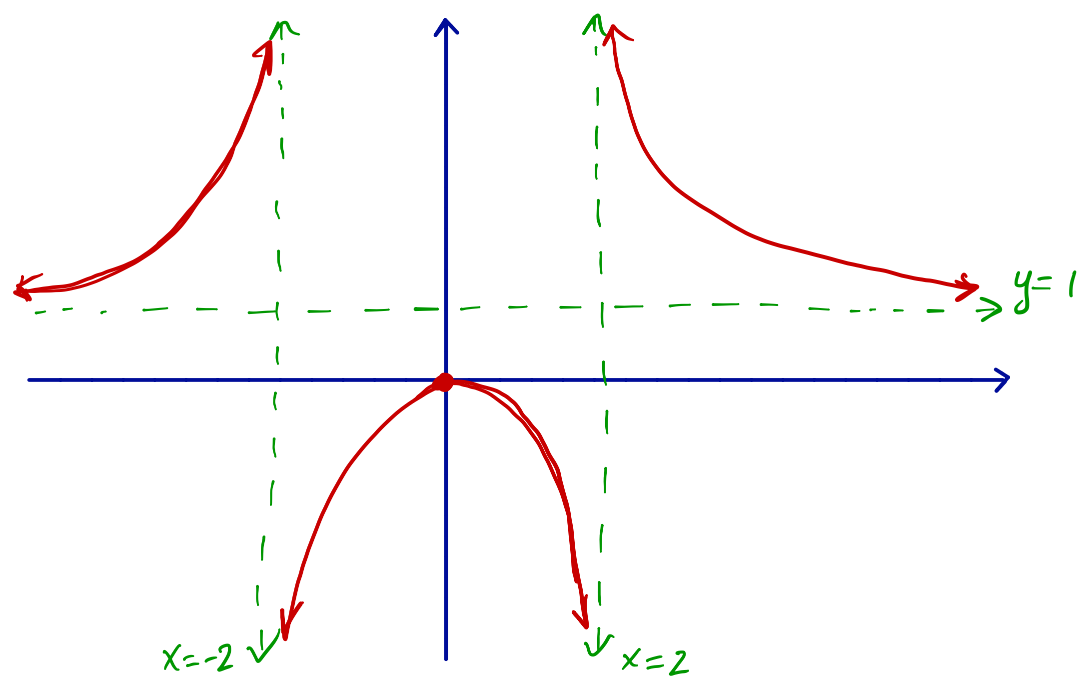
f is concave up on $(-\infty,-2)\cup(2,\infty)$

and concave down on $(-2,2)$

(h) Sketch the curve.

plot important points

$(0,0)$



★ Check your answers using a graphing device!

3. Sketch the graph of $f(x) = x\sqrt{4-x^2}$

(a) Find the domain.

need $4-x^2 \geq 0$. So $-2 \leq x \leq 2$. <u>ANS</u>: $[-2,2]$

(b) Find the x and y-intercepts.

if $x=0$, $y=0$.

If $y=0$, $x=0,+2,-2$.

(c) Find the symmetries/ periodicity of the curve.

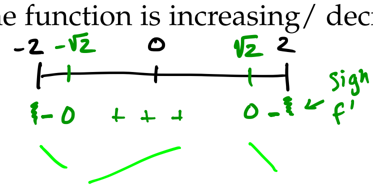
even $\sqrt{4-x^2}$ multiplied by odd x gives odd. $f(x)$ is odd.

(d) Determine the asymptotes.

none

(e,f) Determine where the function is increasing/ decreasing and find the local maximum/ minimum values

$$f'(x) = \frac{2(2-x^2)}{\sqrt{4-x^2}}$$



$f'=0$ when $x=\pm\sqrt{2}$,

f'' undefined at $x=\pm 2$

<u>answer</u>:

f increasing on $(-\sqrt{2},\sqrt{2})$ and decreasing on $(-2,-\sqrt{2})\cup(\sqrt{2},2)$.

f has local min at $x=-\sqrt{2}$, min value -2 and at $x=2$, min value 0.

f has local max at $x=\sqrt{2}$, max value 2 and at $x=-2$, max value 0.

(g) Find the intervals of concavity/inflection points.

$$f''(x) = \frac{2x(6-x^2)}{(4-x^2)^{3/2}}$$

$f''=0$ when $x=0, \cancel{\sqrt{6}}, -\cancel{\sqrt{6}}$ ← not in $[-2,2]$

f'' undefined at $x=-2,2$

$f''<0$ when $x<0$ and $f''>0$ when $x>0$.

<u>answer</u>: f is concave up on $(0,2)$ and concave down on $(-2,0)$.

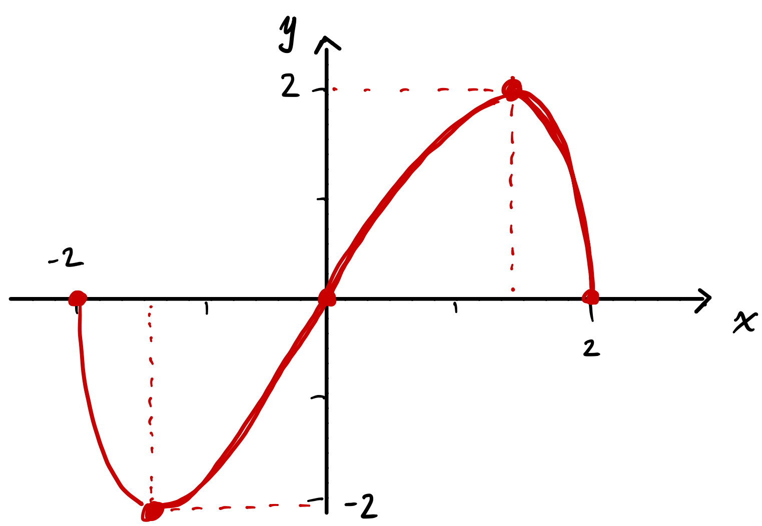
The point $(0,0)$ is an inflection point.

(h) Sketch the curve.

<u>points to plot</u>

$(-2,0)$, $(0,0)$, $(2,0)$

$(-\sqrt{2},-2)$ $(\sqrt{2},2)$



details for example #3

$$f(x) = x(4-x^2)^{1/2}$$

$$f'(x) = 1\cdot(4-x^2)^{1/2} + x\cdot\frac{1}{\cancel{2}}(4-x^2)^{-1/2}\cdot(-\cancel{2}x)$$

$$= (4-x^2)^{1/2} - \frac{x^2}{(4-x^2)^{1/2}} = \frac{4-x^2-x^2}{(4-x^2)^{1/2}} = \frac{2(2-x^2)}{(4-x^2)^{1/2}}$$

get common denominator.

$$f''(x) = \frac{(4-x^2)^{1/2}\cdot 2\cdot(-2x) - 2(2-x^2)\cdot\frac{1}{2}(4-x^2)^{-1/2}(-2x)}{(4-x^2)^1} = \frac{-4x\left[(4-x^2)^{1/2} - \frac{2-x^2}{2(4-x^2)^{1/2}}\right]}{4-x^2}\cdot\frac{2(4-x^2)^{1/2}}{2(4-x^2)^{1/2}}$$

$$8-2x^2-2+x^2 = 6-x^2$$

$$= \frac{-4x\left[\underbrace{2(4-x^2) - (2-x^2)}_{}\right]}{2(4-x^2)^{3/2}} = \frac{-2x(6-x^2)}{(4-x^2)^{3/2}}$$