4-7
(PART 1)

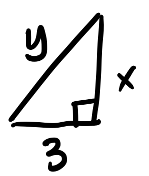
1. Here is a framework for approaching optimization problems.

   (a) Think. Try stuff. These are word problems.

   - draw picture
   - construct some particular examples
   - Identify the goal. maximize or minimize? What quantity?

   - Be willing to try more than one approach.

   (b) Chose notation and explain what it means.

   or

   m = Mary's \$
   t = Tom's \$

   (c) Write the thing you want to maximize or minimize as a function of one variable, including a reasonable domain.

   This needs to be written as a function of 1 variable.

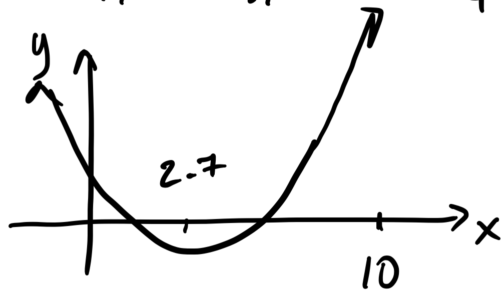
   (d) Use calculus to answer the question.

   - Take derivative.
   - Find crit. #'s.
   - Determine which correspond to an answer.
   - $\boxed{\text{SHOW}}$ your answer is correct!!

2. A Cartoon of Badness

I am supposed to find the maximum of $y=f(x)$ on $[0,10]$. I find $x=2.7$ is the only critical # of f on $[0,10]$. What's wrong with the answer $x=2.7$ as my answer?
" " " $f(2.7)$ as my answer?

What if $f(x)$ looks like →



A MODEL PROBLEM: TWO WAYS Find two positive numbers whose sum is 110 and whose product is a maximum.

thinking: If I am not sure how to begin, I think of specific examples that illustrate the things I am asked about — in this case — #'s that sum to 110 and their products.

Ex's

$1 + 109 = 110$ product: $1 \cdot 109 = 109$

$2 + 108 = 110$ product: $2 \cdot 108 = 216$ ← better! larger product

$10 + 100 = 110$ product: $10 \cdot 100 = 1{,}000$ ← even better!

Set up the general problem:

- Let x, y be positive numbers such that $x + y = 110$ ← Explicitly identify your variables
- $\boxed{\text{maximize the product: } P = xy}$ ← Explicitly identify what quantity is being optimized.
- Using $y = 110 - x$, we have $P(x) = x(110 - x)$ ← write quantity as a function of $\boxed{1}$ variable.
- with $\boxed{\text{domain}}$ $[0, 110]$ since neither x nor y can be negative or larger than 110. ← Identify the domain.

METHOD 1 Closed-Interval Method.

Since $P(x) = 110x - x^2$, $P'(x) = 110 - 2x$.

So critical pts: $x = 55$.

x	55	0	110
$P(x)$	3025	0	0

↑ largest value is maximum

ANSWER: The maximum product is 3025 and occurs when the two numbers are both 55. (That is, when $x = y = 55$.)

actually answer the question.

Method 2 Unique Critical Point Method

Since $P(x) = 110x - x^2$, $P'(x) = 110 - 2x$.

So $P(x)$ has one critical point $x = 55$.

Apply the First Derivative Test:

55

+ + + 0 − − − ← sign of P'

to show that $x = 55$ is a local minimum.

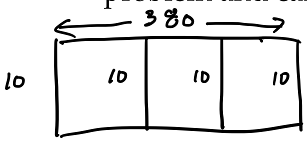
But, $P(x)$ is defined and continuous for all x-values in $[0, 110]$. Thus, the unique local extremum must be absolute.

[Now we draw the same conclusion]

PRACTICE PROBLEMS:

1. A rancher has 800 feet of fencing with which to enclose three adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?

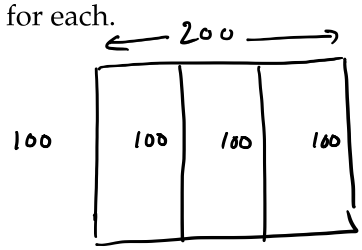
(a) Draw and label with numbers two possible fencing arrangements of the type described in the problem and calculate the enclosed area for each.



$800-40=760$

$\frac{760}{2}=380$

area: 3800 ft^2



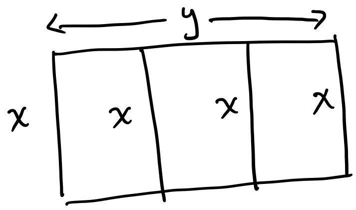
$800-400=400$

$\frac{400}{2}=200$

area: $20{,}000 \text{ ft}^2$

← better. more area.

(b) Draw and label *with appropriate symbols* the general fencing arrangement.



$4x+2y=800$

(c) Write an expression for the total enclosed area using your choice of symbols. Why are you asked to write an expression for *area* and not something else like perimeter or length or volume, etc?

$A = xy$

Why area? Because that is the quantity we are asked to optimize, specifically, maximize.

(d) Write area as a function of *one* variable. Why is this step important? What is the *domain* of your function?

use $4x+2y=800$
or $y=400-2x$
to plug in

$A = xy = x(400-2x)$

ANS: $A(x) = 400x-2x^2$

domain: $[0, 200]$ since neither x nor y can be negative.

(e) Finish the problem by finding the maximum. Show your work in an organized fashion, clearly justifying each step.

closed interval method:

$A'(x) = 400-4x = 0.$

crit pt: $x = 100$

x	0	200	100
$A(x)$	0	0	20,000

← largest is max

Well, that's a fluke that I chose the max.

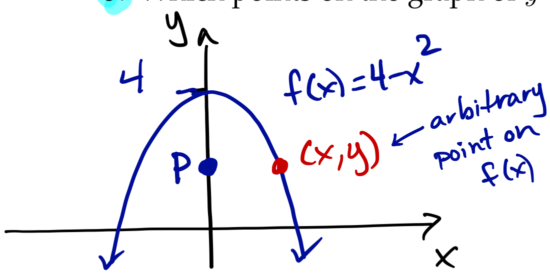
Answer: To maximize area, choose the partitions to have length 100 ft and the remaining side to have length 200 ft.

(f) Is your answer reasonable? Explain.

It does seem like getting closer to a square is good.
I could try some other x-value to check.

Hint: minimize distance squared!!

5. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?



distance from P to (x,y) $= \sqrt{x^2 + (y-2)^2}$, minimize this.

Note: The square root gives me the heebie-jeebies.

So: minimize this instead: $D = x^2 + (y-2)^2$

Substitute in: $y = 4 - x^2$

$$D(x) = x^2 + (4 - x^2 - 2)^2$$
$$= x^2 + (2 - x^2)^2$$

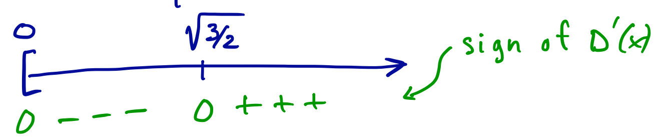
domain: $[0, \infty)$

using symmetry, we know we'll need to pick the "mirror image" of our answer.

First Derivative Test

$$D'(x) = 2x + 2(2 - x^2)(-2x) = 2x(2x^2 - 3)$$

critical points in domain: $x = 2$, $x = \sqrt{3/2}$.



So D has an absolute minimum at $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$.

ANSWER: The points on $y = 4 - x^2$ closest to $(0,2)$ are $\left(\sqrt{3/2}, 5/2\right)$ and $\left(-\sqrt{3/2}, 5/2\right)$.