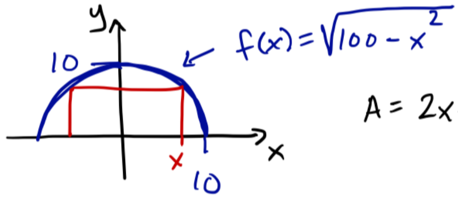
4-7
(PART 2)

1. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 10.
   Hint: The radius r of your circle can be considered a fixed constant. You will expect it to appear in your an



We want to maximize area

$A = 2x\cdot y = 2x\sqrt{100-x^2}$ where $[0,10]$ is the domain

Closed-Interval Test

$$A'(x) = 2\cdot 1\cdot(100-x^2)^{1/2} + \cancel{2}x\cdot\tfrac{1}{\cancel{2}}(100-x^2)^{-1/2}(-2x)$$

$$= 2\sqrt{100-x^2} - \frac{2x^2}{\sqrt{100-x^2}} = \frac{200-2x^2-2x^2}{\sqrt{100-x^2}}$$

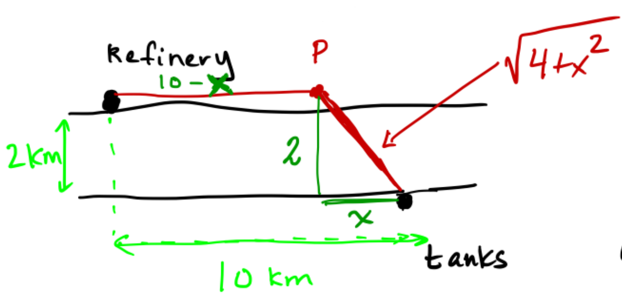
$$= \frac{4(50-x^2)}{\sqrt{100-x^2}}$$

Critical pt: $x=\sqrt{50}=5\sqrt{2}$,
$x=10$

x	0	10	$5\sqrt{2}$
A	0	0	$2\cdot 5\sqrt{2}\cdot\sqrt{50}$ $=200$

Ans: The largest possible rectangle has area 200 units squared.

2. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 10 km downstream of the refinery. The cost of laying pipe is \$ 10,000/ km over land to a point P on the opposite bank and then \$ 40,000/km under the river to the tanks. To minimize the cost of pipeline, where should P be located?



Cost (in 10,000's)

$C(x) = 1\cdot(10-x) + 4\cdot(4+x^2)^{1/2}$

where x can take values $[0,10]$.

$$C'(x) = -1 + 4\cdot\tfrac{1}{2}(4+x^2)^{-1/2}(2x) = \frac{4x}{\sqrt{4+x^2}} - 1$$

Critical points: $4x=(4+x^2)^{1/2}$ or $16x^2=4+x^2$. Thus, $x^2=4/15$. So $x=2/\sqrt{15}$.

x	0	10	$2/\sqrt{15}$
C	$10+8$ $=18$	$4\sqrt{104}$ $\approx 4\sqrt{26}$	17.45 ← min

Ans:
P should be located $10-\frac{2}{\sqrt{15}}$ km down river from the refinery.

3. Two posts, on 12 feet high and the other 28 feet high stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?

$$L(x) = (12^2 + x^2)^{1/2} + \left((30-x)^2 + 28^2\right)^{1/2} \quad ; \text{ domain: } [0, 30]$$

$$L'(x) = \cancel{\tfrac{1}{2}}(144 + x^2)^{-1/2}(\cancel{2}x) + \cancel{\tfrac{1}{2}}\left((30-x)^2 + 28^2\right)^{-1/2}(\cancel{2}(30-x)(-1))$$

$$= \frac{x}{\sqrt{144 + x^2}} - \frac{30 - x}{\sqrt{(30-x)^2 + 28^2}}$$

− 0 + + + ← sign of L'

0 9 30

critical pts:

$$x\left((30-x)^2 + 28^2\right)^{1/2} = (30-x)(144 + x^2)^{1/2}$$

$$x^2(900 - 60x + x^2 + 784) = (900 - 60x + x^2)(144 + x^2)$$

$$x^4 - 60x^3 + 1684x^2 = x^4 - 60x^3 + 1044x^2 - 8640x + 129600$$

$$640x^2 + 8640x - 129600 = 0 \quad \div 320$$

$$2x^2 + 27x - 405 = 0$$

$$x = 9$$

Ans:

Length is minimized if the wire is fixed 9 ft from the 12 ft post.

4. Four feet of wire is used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum area?

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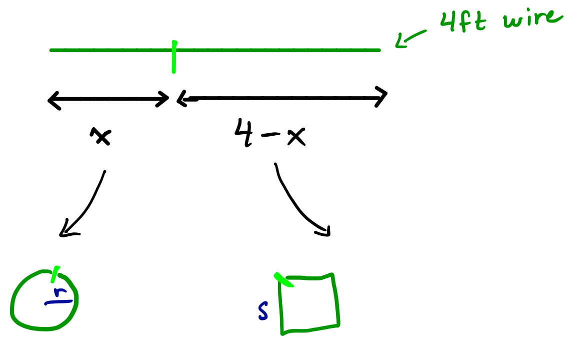
5. Four feet of wire is used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum area?



circumference is x

So:

$x = 2\pi r$ or

$r = \frac{x}{2\pi}$.

Thus area of circle $= \pi\left(\frac{x}{2\pi}\right)^2$

$= \frac{x^2}{4\pi}$

perimeter is $4-x$

So:

$4 - x = 4s$ or

$s = \frac{4-x}{4}$.

Thus, area square $= \frac{1}{16}(4-x)^2$

area enclose $= A(x) =$ area in circle $+$ area in square

$= \frac{1}{4\pi}\cdot x^2 + \frac{1}{16}(4-x)^2$; domain: $[0,4]$.

Now, $A'(x) = \frac{x}{2\pi} + \frac{1}{8}(4-x)(-1) = \left(\frac{x}{2\pi} - \frac{4-x}{8} = 0\right)\cdot 8\pi$

Critical pts: $4x - 4\pi + \pi x = 0$ or $x = \frac{4\pi}{4+\pi} \approx 1.76$

$- - - 0 + + +$ ← sign A'

0   1.76   4

↖ location of local min.

$x = 0$, $A(0) = 1$; $x = 4$, $A(4) = \frac{4}{\pi} > 1$.

So, area is maximized when all of the wire is used to make the circle