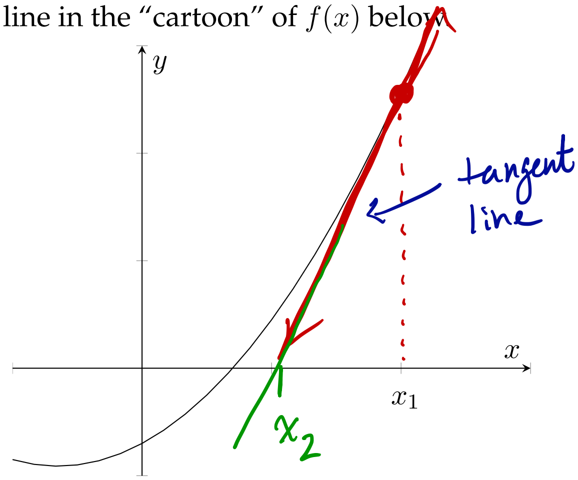
LECTURE NOTES: 4-8 NEWTON'S METHOD (PART 1)

MOTIVATING QUESTION: Recall that we wanted to find the x-intercepts of $f(x) = x - 2\sin x$. From the graph we knew there exists a positive (and negative) solution. How to find it?

WARM-UP PROBLEMS:

1. Write the equation of the line tangent to the curve $y = f(x)$ at the x-value x_1. Sketch the tangent line in the "cartoon" of $f(x)$ below.



$m = f'(x_1)$ slope

$(x_1, f(x_1))$ point

line: $y - f(x_1) = f'(x_1)(x - x_1)$

or $\boxed{y = f'(x_1)(x-x_1) + f(x_1)}$

2. In your picture above, label the x-value where the tangent line intersects the x-axis as x_2.

3. Solve for x_2 using your equation from part (1) above.

x_2 occurs when $y=0$. Set $y=0$ & solve for x. here

$0 = f'(x_1)(\boxed{x} - x_1) + f(x_1)$

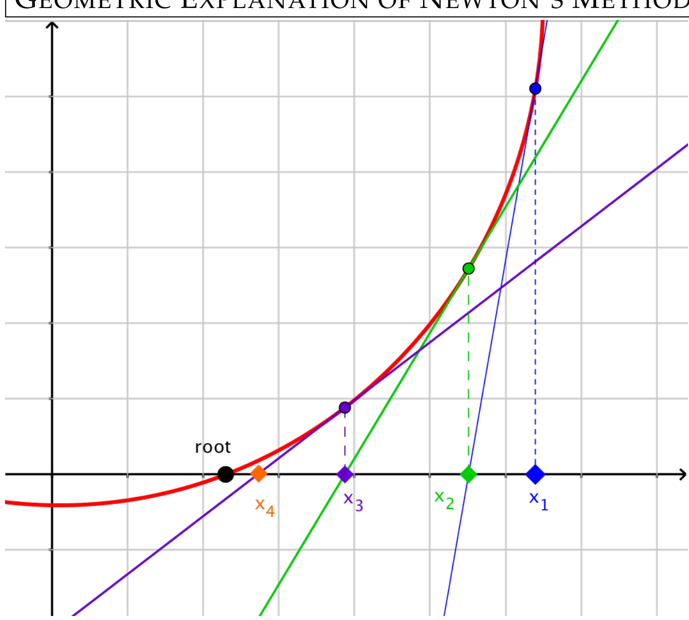
$-f(x_1) = f'(x_1)\boxed{x} - x_1 f'(x_1)$

$x_1 f'(x_1) - f(x_1) = f'(x_1) \cdot \boxed{x}$

$\boxed{x_2} = \dfrac{x_1 f'(x_1) - f(x_1)}{f'(x_1)} = \underbrace{x_1 - \dfrac{f(x_1)}{f'(x_1)}}$

observe, this is actually a really simple expression.

GEOMETRIC EXPLANATION OF NEWTON'S METHOD:



Algebraically

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

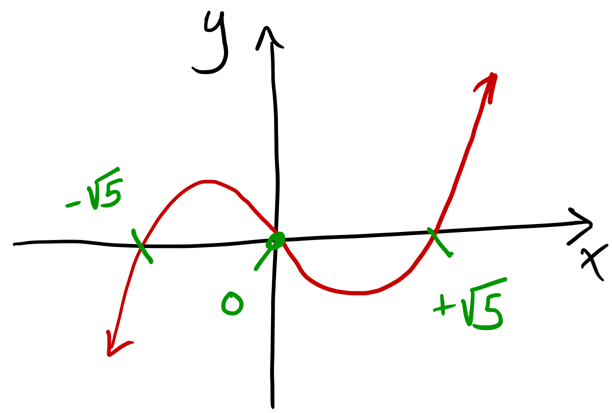
* There is nothing special about $x_1 \to x_2$.

MODEL PROBLEM: Let $f(x) = x^3 - 5x$

1. Factor $f(x)$, find its roots algebraically, and sketch its graph.

$f(x) = x(x^2-5) = x(x+\sqrt{5})(x-\sqrt{5})$

roots $x = 0, -\sqrt{5}, +\sqrt{5}$



Can you draw a picture of the calculations you just performed!?

2. Assume you couldn't factor the function and you wanted to find its positive root. What would be a reasonable first guess and why?

$x=3, \; f(3) = 27-15 = 12 > 0$

$x=2, \; f(2) = 8-10 = -2 < 0$

So there's a positive root between $x=2$ and $x=3$.

I'd pick $\boxed{x=2.5}$.

3. Using a first guess of $x_1 = 3$, calculate 2 iterations of Newton's method

Recall $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $f(x) = x^3-5x$, $f'(x) = 3x^2-5$

If $x_1 = 3$, then $x_2 = 3 - \dfrac{f(3)}{f'(3)} = 3 - \dfrac{27-15}{27-5} = 3 - \dfrac{12}{22} \approx 2.45454545$ 1 iteration

If $x_2 = 2.45454545$, then $x_3 = (2.45454545) - \dfrac{f(2.45454545)}{f'(2.45454545)} \approx 2.26215377$

4. How close is your estimate of the root, x_3, to the actual root?

$\sqrt{5} \approx 2.23606798$. Difference: less than 0.0260

not too shabby.

QUESTIONS FOR YOU:

1. In general, when applying Newton's Method can there exist bad choices for x_1? Explain.

   - bad if $f'(x_i) = 0$.
   - may converge to a different root.

2. Use Newton's Method to approximate any zeros of $g(x) = x - 2\sin x$ accurate to at least 9 decimal places. [NOTE: The FIRST thing you must do is make a guess at where that root might be. Make it a GOOD guess.]

Clearly $x=0$ is a zero of $g(x)$. Also, since $g(x)$ is an odd function, we know that if we find a positive root we will find the symmetric negative root simultaneously.

$g(\frac{\pi}{2}) = \frac{\pi}{2} - 2\sin(\frac{\pi}{2}) = \frac{\pi}{2} - 2 < 0$

$g(\pi) = \pi - 2\sin(\pi) = \pi > 0$

So there is a root between 1.5 and 3.

Pick $x_1 = 2$. Newton's Formula here is:

$$x_{n+1} = x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n}$$

$x_1 = 2$

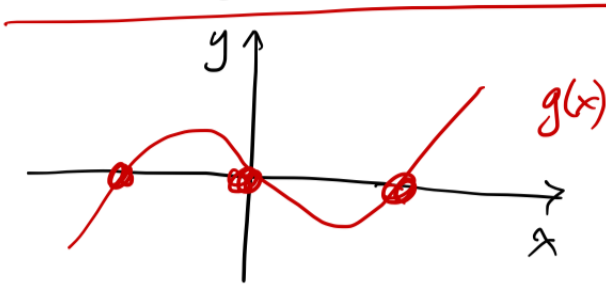
$$x_2 = 2 - \frac{2 - 2\sin(2)}{1 - 2\cos(2)} \approx 1.90099594$$

$x_3 \approx 1.895511645$

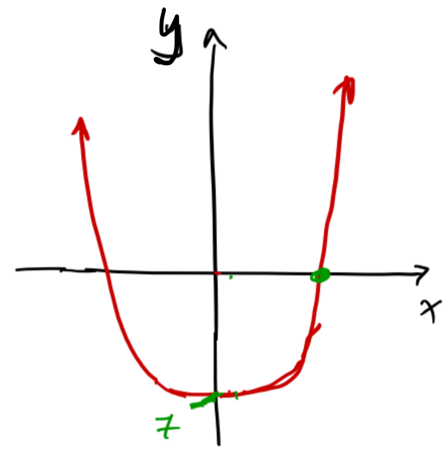
$x_4 \approx 1.895494267$

$x_5 = x_4$.

$\boxed{\text{So } x = \pm 1.895494267, 0 \text{ are zeros of } g(x).}$



3. Estimate $\sqrt[6]{7}$ correct to 5 decimal places. [Note: *you* must construct an appropriate $f(x)$ here.]



$f(x) = x^6 - 7$ will have $\sqrt[6]{7}$ as a root

$f(1) = -6 < 0$, $f(2) > 0$. I'll pick $x_1 = 1.1$

Newton's Formula would be:

$$x_{n+1} = x_n - \frac{(x_n)^6 - 7}{6x_n^5} = x_n - \frac{1}{6}x_n + \frac{7}{6}x_n^{-5}$$

$$= \frac{1}{6}\left(5x_n + 7x_n^{-5}\right)$$

If $x_1 = 1.1$, then $x_2 = \frac{1}{6}\left(5(1.1) + 7(1.1)^{-5}\right)$

≈ 1.641074877

$x_3 \approx 1.465580165$

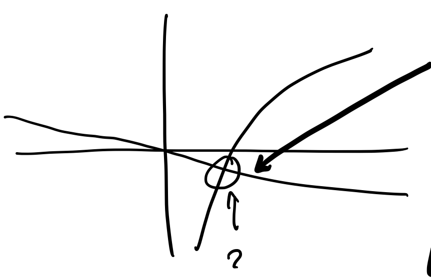
$x_4 \approx 1.39386042$

$x_5 \approx 1.383293572$

$x_6 \approx \underline{1.38308}7631$

$x_7 \approx \underline{1.383087}554$

4. Explain how Newton's Method could be used to find points of intersections between curves, say where $f(x) = -x/3$ and $g(x) = \ln x$ intersect.



means $-\frac{x}{3} = \ln x$ or

$f(x) = \ln x + \frac{x}{3} = 0$

Use Newton's Method to find a root of $f(x)$.