1. Sketch a graph that satisfies all of the conditions:

   domain $f = (-\infty, \infty)$,
   $f(3) = -1, \quad f'(3) = 0$
   $f'(x) < 0$ when $x < 3, f'(x) > 0$ when $x > 3$,
   $f''(x) < 0$ when $x < 0, \quad f''(x) > 0$ when $x > 0$
   $\lim_{x\to-\infty} f(x) = 4$

2. Evaluate the following limits.

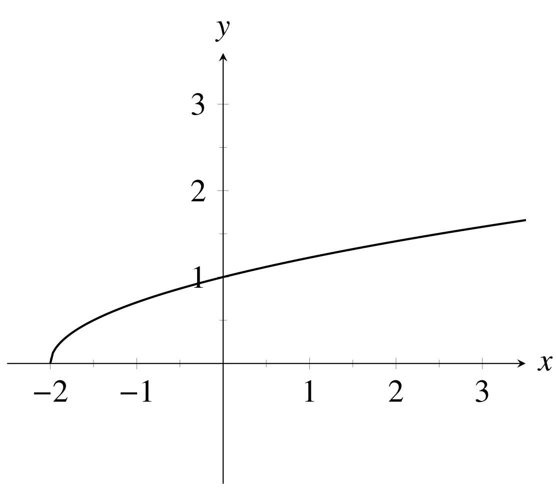
   (a) $\lim_{x\to 0} \frac{\sin(x^2)}{x^2}$
   (b) $\lim_{x\to 0^+} \sqrt{x}\ln(x)$

3. A function and its first and second derivatives are given below.

$$f(x) = x^{5/3} - 5x^{2/3}, \qquad f'(x) = \frac{5x-10}{3x^{1/3}}, \qquad f''(x) = \frac{10x+10}{9x^{4/3}}$$

   (a) Identify any critical points of $f(x)$.
   (b) Find the intervals of increase and decrease, and identify the locations of any local maximum or minimum values.
   (c) Find the intervals of concavity and the $x$-values of any inflection points.

4. The graph of the function $f(x) = \sqrt{\frac{x}{2} + 1}$ is shown.




   (a) Let $G(x)$ be the square of the distance from the origin to a point on the graph of $y = f(x)$. Write an expression for $G(x)$.
   (b) Use the expression for $G(x)$ to find the closest point on the graph $y = f(x)$ to the origin.
   (c) Show your result by adding a point, with coordinates, to the graph.

5. A ship passes a lighthouse at 3:30pm, sailing to the east at 5 mph, while another ship sailing due south at 6 mph passes the same point half an hour later. How fast will the distance between the ships be increasing at 5:30pm?

6. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.1 cm thick to a hemispherical dome with radius 10 m. Give your final answer with proper units. (Note the volume of a sphere is $V = \frac{4}{3}\pi r^3$.)

7. Find the linearization of $f(x) = e^x$ at $a = 0$ and use it to estimate $e^{0.1}$.

8. Solve the initial value problem. If the velocity of an object is given by $v(t) = e^t + t$, find the position of the object assuming that the initial position of the object is 0. (That is, $s(0) = 0$.)

9. Evaluate the indefinite integral below. Give the most complete answer. $\int (5\sec^2(x) + \frac{1}{x^5})\,dx$.

10. Estimate the area under the curve $f(x) = x^3$ and above the $x$-axis on the interval $[0, 2]$ using 4 rectangles and right-hand endpoints. (i.e. Find $R_4$.)

11. Determine the absolute maximum and absolute minimum of $f(t) = \frac{\sqrt{t}}{1+t^2}$ on the interval $[0, 2]$.

8. $S(t) = \int v(t)\,dt = \int (e^t + t)\,dt = e^t + \frac{1}{2}t^2 + C.$

$0 = S(0) = e^0 + \frac{1}{2}0^2 + C = 1 + C$. So $C = -1$

$\boxed{S(t) = e^t + \frac{1}{2}t^2 - 1}$

9. $\int (5\sec^2 x + x^{-5})\,dx = \boxed{5\tan(x) - \frac{1}{4}x^{-4} + C}$

most general answer.

10. $R_4 = \frac{2}{4}\left(f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2)\right)$

$= \frac{1}{2}\left(\frac{1}{8} + 1 + \frac{27}{8} + 8\right) = \frac{1}{2}\left(\frac{50}{4}\right) = \boxed{\frac{25}{4}}$

1

9. **(10 points)**

Sketch a graph that satisfies all of the conditions:

domain $f = (-\infty, \infty)$,

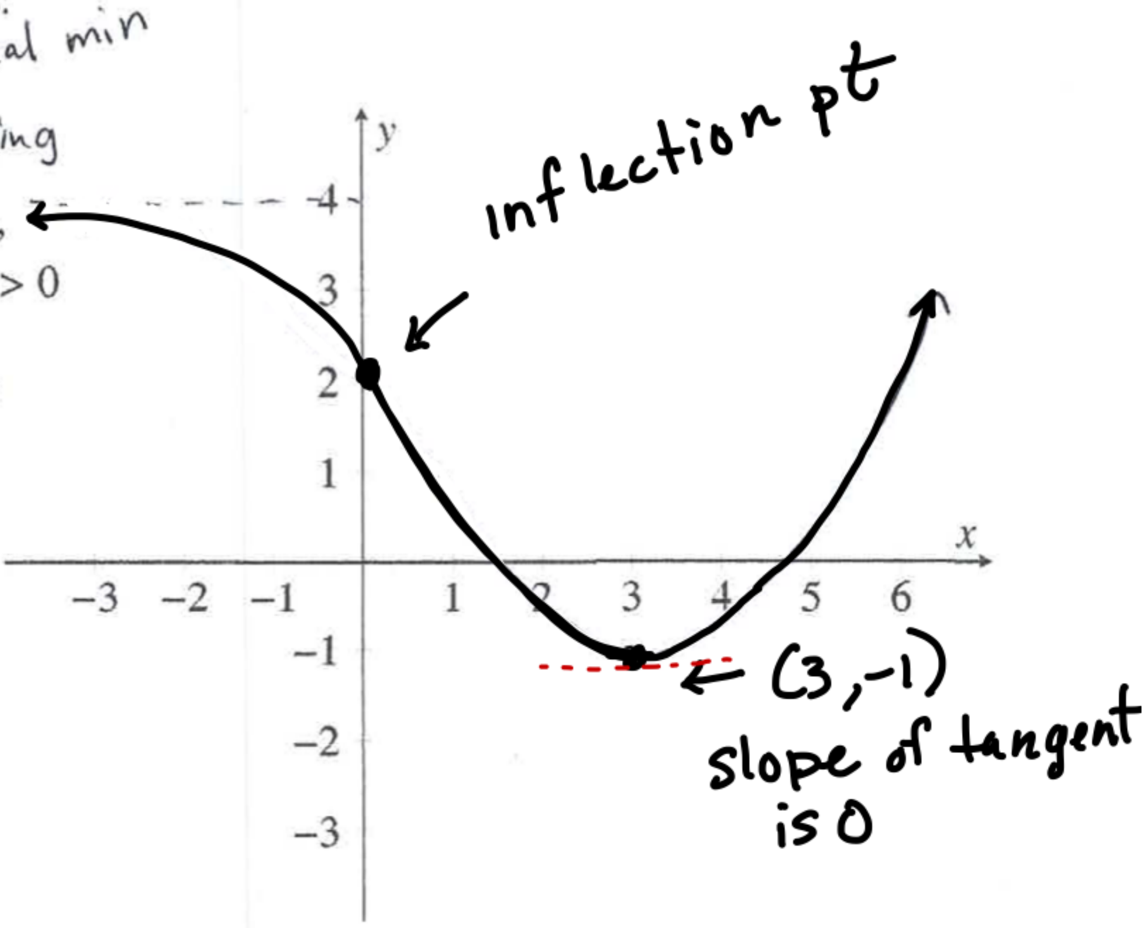
$f(3) = -1$, $f'(3) = 0$ — critical pt and local min

decreasing — $f'(x) < 0$ when $x < 3$, $f'(x) > 0$ when $x > 3$, — increasing

$f''(x) < 0$ when $x < 0$, $f''(x) > 0$ when $x > 0$

$\lim_{x \to -\infty} f(x) = 4$

cc down — horizontal asymptote — concave up

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7. **(10 points)**

Evaluate the following limits. [Note: You should be careful to apply L'Hôpital's rule **only** when appropriate.]

**a.** $\lim_{t \to 0} \frac{\sin(t^2)}{t^2}$ $\overset{\frac{0}{0}}{\underset{L'H.}{=}}$ $\lim_{t \to 0} \frac{\cos(t^2) \cdot \cancel{2t}}{\cancel{2t}} = \lim_{t \to 0} \cos(t^2)$

$= 1$

**b.** $\lim_{x \to 0^+} \sqrt{x} \ln(x) \overset{0 \cdot \infty}{=} \lim_{x \to 0^+} \frac{\ln x}{x^{-1/2}} \overset{\frac{\infty}{\infty}}{\underset{L'H.}{=}} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-3/2}}$

$= \lim_{x \to 0^+} -2x^{1/2} = 0$

$\boxed{3}$

**8. (10 points)**

A function and its first and second derivatives are given below.

$$f(x) = x^{5/3} - 5x^{2/3}, \qquad f'(x) = \frac{5x - 10}{3x^{1/3}}, \qquad f''(x) = \frac{10x + 10}{9x^{4/3}}$$

**a.** Find the intervals of increase and decrease, and identify the locations of any local maximum or minimum values.

$f'(x) = 0$ when $5x - 10 = 0$

$5x = 10$

$x = 2$

and undefined when $x = 0$

sign of $f'$: $+$ $-$ $+$

$-1 \quad 0 \quad 1 \quad 2 \quad 3$

$f$ is increasing on the interval $(-\infty, 0) \cup (2, \infty)$

and decreasing on the interval $(0, 2)$

$f$ has a local max at $x = 0$ and a local min at $x = 2$

**b.** Find the intervals of concavity and the $x$-values of any inflection points.

$f''(x) = 0$ when $x = -1$

and undefined at $x = 0$

Sign of $f''$: $-$ $+$ $+$

$-2 \quad -1 \quad -\frac{1}{2} \quad 0 \quad 1$

$f$ is concave up on $(-1, \infty)$

and concave down on $(-\infty, -1)$

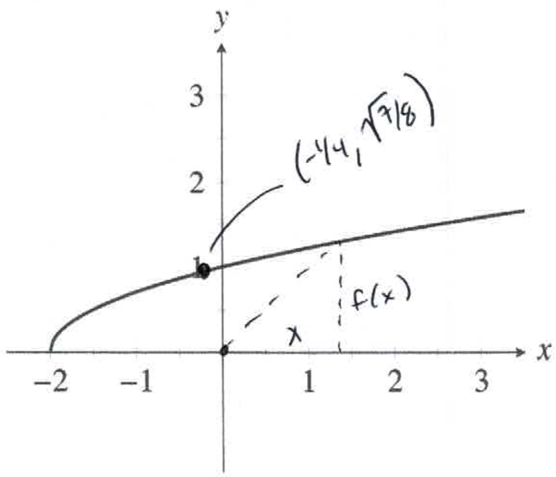
$f$ has an inflection point at $x = -1$.

9. (10 points)

4

**10. (12 points)**

The graph of the function $f(x) = \sqrt{\frac{x}{2} + 1}$ is shown.




**a.** Let $G(x)$ be the square of the distance from the origin to a point on the graph of $y = f(x)$. Write an expression for $G(x)$.

$$x^2 + f(x)^2 = G(x)$$

$$G(x) = x^2 + \left(\sqrt{\frac{x}{2}+1}\right)^2$$

$$G(x) = x^2 + \frac{x}{2} + 1$$

**b.** Use the expression for $G(x)$ to find the closest point on the graph $y = f(x)$ to the origin.

$G'(x) = 2x + \frac{1}{2}$ $\quad$ $G'(x) = 0$ when $x = -1/4$

sign of $G'(x)$ $\quad$ $-$ | $+$ at $-1/4$ $\quad$ $G(x)$ has a min at $x = -1/4$

$$f(-1/4) = \sqrt{\frac{-1/4}{2}+1} = \sqrt{7/8}$$

Closest point is $(-1/4, \sqrt{7/8})$

**c.** Show your result by adding a point, with coordinates, to the graph.

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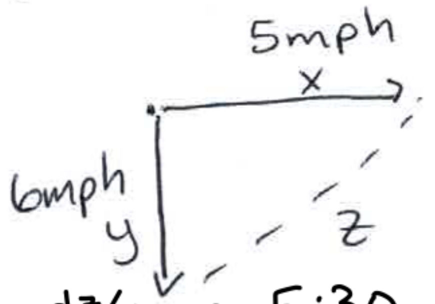
**5. (12 points)**

A ship passes a lighthouse at 3:30pm, sailing to the east at 5 mph, while another ship sailing due south at 6 mph passes the same point half an hour later. How fast will the distance between the ships be increasing at 5:30pm?




Want: $dz/dt$ @ 5:30pm.

at 5:30pm

$x = 10$ $\quad \frac{dx}{dt} = 5$

$y = 9$ $\quad \frac{dy}{dt} = 6$

$10^2 + 9^2 = z^2$

$\sqrt{181} = z$

$$x^2 + y^2 = z^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2z}$$

$$\frac{dz}{dt} = \frac{\cancel{2}(10)\cdot 5 + \cancel{2}(9)(6)}{\cancel{2}\sqrt{181}}$$

$$\frac{dz}{dt} = \frac{104}{\sqrt{181}} \text{ mph}$$

$$\begin{array}{r} 50 \\ +54 \\ \hline 104 \end{array}$$

7

14. (6 points) Find the linearization of $f(x) = e^x$ at $a = 0$ and use it to estimate $e^{0.1}$

$f'(x) = e^x \qquad a=0, \; f(0)=1$

$m = f'(0) = 1$

$y - 1 = 1(x - 0)$

$\boxed{y = x + 1}$

$e^{0.1} \approx 0.1 + 1$

$= \boxed{1.1}$

6

16. (6 points) Use differentials to estimate the amount of paint needed to apply a coat of paint 0.1 **cm** thick to a hemispherical dome with radius 10 **m**. Give your final answer with proper units. (The volume of a sphere is $V = \frac{4}{3}\pi r^3$) hemisphere – half

$V = \frac{2}{3}\pi r^3$

$dV = 2\pi r^2 dr$

$0.1 \text{ cm} = 0.1 \div 100 \text{ m}$

$= 0.001 \text{ m}$

OR $10 \text{ m} = 10 * 100 \text{ cm}$

$= 1000 \text{ cm}$

$dV = 2\pi(10^2)(0.001)$

$= 200\pi(0.001)$

$= \boxed{0.2\pi \text{ m}^3}$



NL

(11.) $f(t) = \frac{\sqrt{t}}{1+t^2}$ on $[0,2]$

$$f'(t) = \frac{(1+t^2)(\frac{1}{2}t^{-1/2}) - (t^{1/2})(2t)}{(1+t^2)^2} = \frac{\frac{1+t^2}{2\sqrt{t}} - \frac{2t^{3/2}}{1}}{(1+t^2)^2} \frac{\cdot 2\sqrt{t}}{\cdot 2\sqrt{t}}$$

$$= \frac{1+t^2-2t^2}{2\sqrt{t}(1+t^2)^2} = \frac{1-t^2}{2\sqrt{t}(1+t^2)^2}$$

$f'(t) = 0$ when $t = \pm 1$. Only $t=1$ is in the interval $[0,2]$

Chart

| $t$ | 0 | 2 | 1 |
|---|---|---|---|
| $f(t)$ | 0 | $\frac{\sqrt{2}}{5}$ | $\frac{1}{2}$ |

smallest  largest

answer: minimum : $y=0$

maximum : $y = \frac{1}{2}$