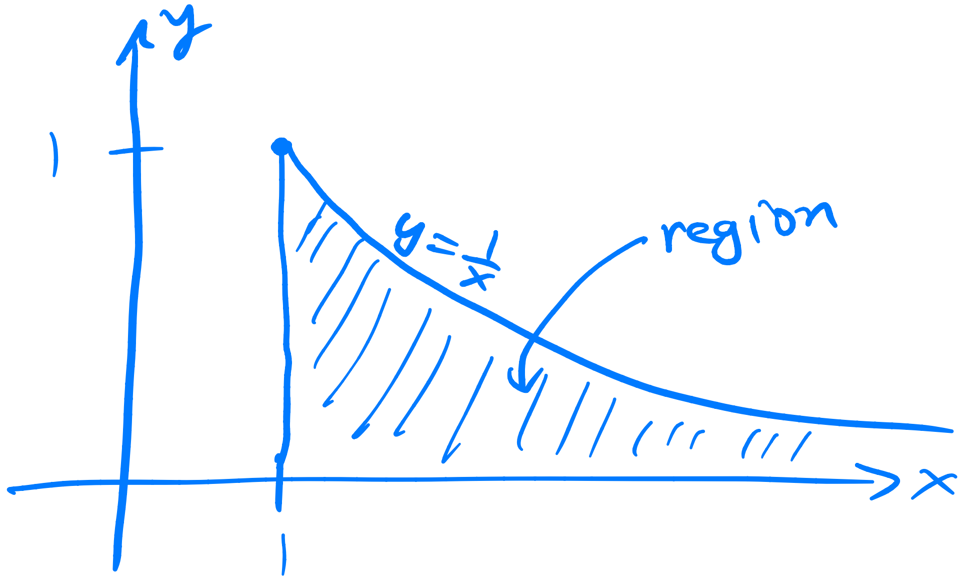
Name: SOLUTIONS

Math 252 Calculus 2 (Bueler) Tuesday, 13 December 2022

Final Exam

No book, electronics, calculator, or internet access. 125 points possible. 125 minutes maximum. Allowed notes: 1/2 sheet of letter paper (i.e. half of 8.5×11 sheet), with anything written on both sides.

1. **(a)** (*5 pts*) Sketch the region bounded by $y = \frac{1}{x}$ and the x-axis, on the interval $1 \le x < +\infty$.



(b) (*8 pts*) Compute the volume of the solid of revolution found by rotating the region in **(a)** around the x-axis, or show that it is infinite.

$$V = \int_1^\infty \pi y(x)^2 dx = \pi \int_1^\infty \frac{dx}{x^2}$$

$$= \pi \lim_{t\to\infty} \int_1^t x^{-2} dx = \pi \lim_{t\to\infty} \left[-x^{-1}\right]_1^t$$

$$= \pi \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = \pi(0+1) = \pi$$

(converges)

2. (*7 pts*) Set up, but **do not evaluate**, an integral for the arclength of the parametric curve defined by $x(t) = e^t$, $y(t) = \ln t$ on the interval $1 \leq t \leq 10$.

$$L = \int_1^{10} \sqrt{(e^t)^2 + \left(\frac{1}{t}\right)^2}\, dt$$

$$\left(\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = \frac{1}{t} \right)$$

3. (*8 pts*) Compute the indefinite integral:

$$\int x\, 5^{x^2}\, dx = \int 5^u \frac{du}{2} = \frac{1}{2} \frac{5^u}{\ln 5} + C$$

$$\begin{bmatrix} u = x^2 \\ du = 2x\,dx \end{bmatrix}$$

$$= \frac{5^{x^2}}{2 \ln 5} + C$$

4. Evaluate the integrals:

(a) (*6 pts*) $\int_0^3 x\,e^{-x}\,dx =$ $x(-e^{-x})\Big]_0^3 - \int_0^3 (-e^{-x})\,dx$

[by parts: $u = x$ $\quad v = -e^{-x}$
$du = dx$ $\quad dv = e^{-x}dx$]

$= -\left[xe^{-x}\right]_0^3 + \int_0^3 e^{-x}\,dx = -(3e^{-3} - 0) - \left[e^{-x}\right]_0^3$

$= -3e^{-3} - e^{-3} + e^{-0} = 1 - 4e^{-3}$

(b) (*6 pts*) $\int \frac{dx}{x(1+x)} =$ $\int \frac{1}{x} - \frac{1}{1+x}\,dx$

partial fractions:

$\frac{1}{x(1+x)} = \frac{A}{x} + \frac{B}{1+x}$

$0x + 1 = A(1+x) + Bx$

$= (A+B)x + A$

$A = 1, \quad B = -1$

$= \ln|x| - \ln|1+x| + C$

$= \ln\left|\frac{x}{1+x}\right| + C$

5. Evaluate the indefinite integrals:

(a) (*6 pts*) $\int \cos^2\theta \sin^2\theta \, d\theta = \frac{1}{4}\int (1+\cos 2\theta)(1-\cos 2\theta)\, d\theta$

$$= \frac{1}{4}\int 1-\cos^2(2\theta)\, d\theta = \frac{1}{4}\int 1-\frac{1}{2}(1+\cos(4\theta))\, d\theta$$

$$= \frac{1}{8}\int 1-\cos(4\theta)\, d\theta = \frac{1}{8}\left[\theta - \frac{\sin(4\theta)}{4}\right]+C$$

$$= \frac{\theta}{8} - \frac{\sin(4\theta)}{32} + C$$

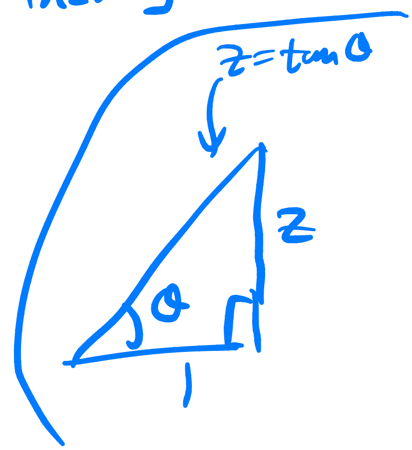
(b) (*6 pts*) $\int \frac{dz}{\sqrt{1+z^2}} = \int \frac{\sec^2\theta \, d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta}{\sec\theta}\, d\theta$

$$\left[\begin{aligned} z &= \tan\theta \\ dz &= \sec^2\theta \end{aligned}\right]$$

-2 for stopping here

$$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

[from memory of "trick"]



$$= \ln\left|\sqrt{1+z^2} + z\right| + C$$

6. Determine whether the following series converge or diverge. Explain your reasoning and identify any test used.

(a) (*6 pts*) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

$b_n = \frac{1}{\sqrt{n+3}} > 0$, b_n decreasing,

$\lim_{n\to\infty} b_n = 0$

Converges

by A.S.T.

(b) (*6 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10+\sqrt{n}}$

divergence test:

$$\lim_{n\to\infty} \frac{n}{10+\sqrt{n}} = \lim_{n\to\infty} \frac{n \cdot \frac{1}{\sqrt{n}}}{(10+\sqrt{n})\cdot\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\frac{10}{\sqrt{n}}+1} = \frac{\infty}{0+1} = +\infty$$

$\therefore$ diverges

or

limit comparison: compare to $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n}} = \sum_{n=1}^{\infty} n^{1/2}$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{-1/2}}$$

$$\lim_{n\to\infty} \frac{\frac{n}{10+\sqrt{n}}}{\frac{n}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n(\sqrt{n})}{(10+\sqrt{n})n}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}+10n} = 1 \neq 0, \infty$$

p-series with $p = -\frac{1}{2} \leq 1$ diverges

7. **(a)** (*8 pts*) Find the Maclaurin series by any convenient method:

$f(x) = x^2 e^{-x^2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \underline{\text{so}} \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$\underline{\text{so}}$

$$f(x) = x^2 e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{(n-1)!} \Bigg\} \text{just as good!}$$

(b) (*8 pts*) Use the result in **(a)** to compute the following antiderivative as a power series:

$$\int x^2 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n+2}\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+3}}{2n+3}$$

$$= C + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} \frac{x^{2n+1}}{2n+1} \Bigg\} \text{just as good}$$

8. **(a)** (*6 pts*) Does the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge or diverge? Explain your reasoning and identify any test used.

ratio test:

$$\rho = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{n+1}{n} \stackrel{L'H}{=} \frac{1}{2} < 1 \quad \therefore \text{converges}$$

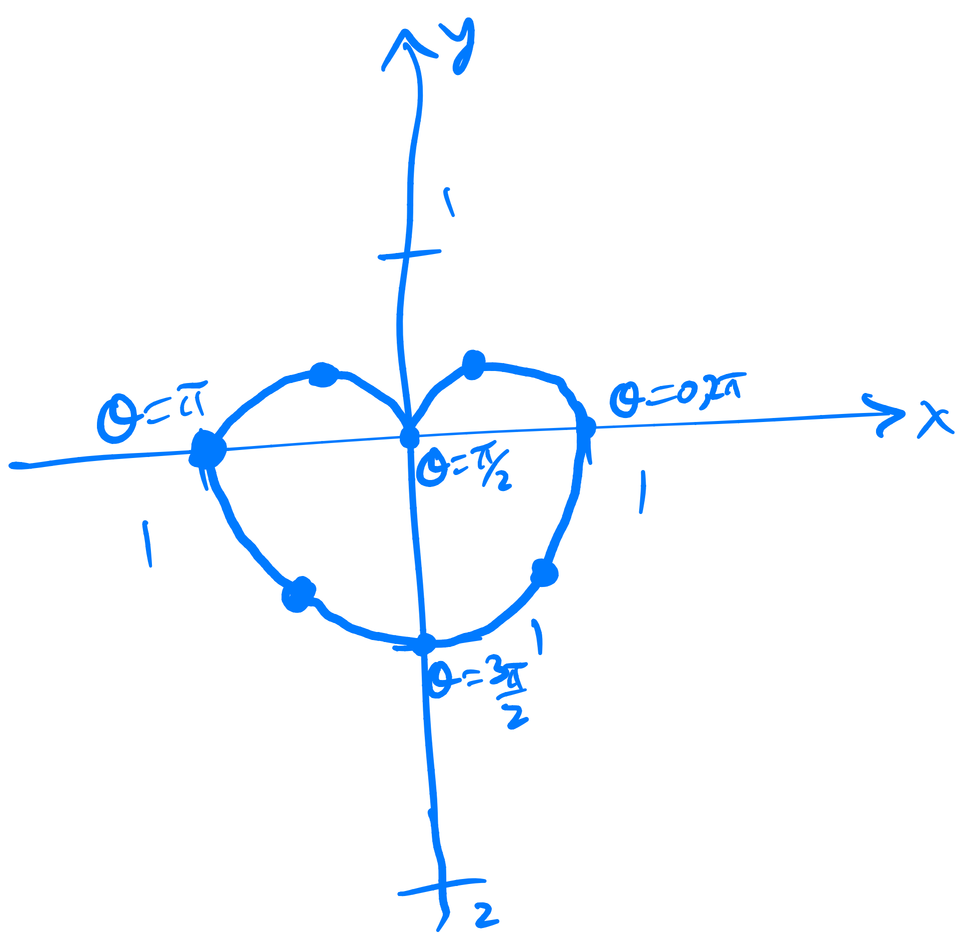
(root test similar)

(b) (*5 pts*) Compute and simplify S_3 for the series in part **(a)**.

$$S_3 = \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} = \frac{11}{8}$$

9. (*6 pts*) Make a careful and reasonably-large sketch of the cardiod $r = 1 - \sin\theta$. (*Label the axes and give dimensions/values along the axes.*)

r	θ
0	1
$\pi/4$	$1-\frac{1}{\sqrt{2}} \approx 0.3$
$\pi/2$	0
$3\pi/4$	$1-\frac{1}{\sqrt{2}} \approx 0.3$
π	1
$5\pi/4$	$1+\frac{1}{\sqrt{2}} \approx 1.7$
$3\pi/2$	2
$7\pi/4$	$1+\frac{1}{\sqrt{2}} \approx 1.7$
2π	1



10. Consider the parametric curve $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$.

(a) (*6 pts*) Find the equation of the tangent line at $t = 1$.

$$m = \left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \left.\frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}}\right|_{t=1} = \text{"}\frac{2}{0}\text{"}$$

$$\therefore \quad m = +\infty \quad \text{(vertical)}$$

$$x(1) = 1 + \frac{1}{1} = 2$$

$$\therefore \quad x = 2$$

(b) (*6 pts*) Compute the second derivative $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}} = \frac{t^2+1}{t^2-1}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{t^2+1}{t^2-1}\right)}{dx/dt} = \frac{\frac{2t(t^2-1)-(t^2+1)(2t)}{(t^2-1)^2}}{1-\frac{1}{t^2}}$$

$$= \frac{2t(t^2-1-t^2-1)}{(t^2-1)^2\left(1-\frac{1}{t^2}\right)} = \frac{2t(-2)}{(t^2-1)^2(t^2-1)\frac{1}{t^2}}$$

$$= \frac{-4t^3}{(t^2-1)^3}$$

11. (*8 pts*) Find the radius and interval of convergence of the power series:
$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n^2}$$

root test: $\rho = \lim_{n\to\infty} \frac{|x+5|}{(\sqrt[n]{n})^2} = \frac{|x+5|}{1} = |x+5| < 1$

$|x+5|<1 \iff -1 < x+5 < 1 \iff -6 < x < -4$

$x=-6$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges (absolutely) $p=2$

$x=-4$: $\sum_{n=1}^{\infty} \frac{1^n}{n^2}$ converges $p=2$

$\therefore$ $I = [-6,-4]$ and $R=1$

12. (*8 pts*) Find the general solution to the differential equation $y' = \ln x + \tan x$.

$y(x) = \int \ln x + \tan x \, dx = \int \ln x \, dx + \int \frac{\sin x}{\cos x} dx$

$= x\ln x - \int x \cdot \frac{1}{x} dx - \ln|\cos x| + C$

$\left[\begin{array}{ll} u=\ln x & v=x \\ du=\frac{1}{x}dx & dv=dx \end{array}\right]$

$= x\ln x - x - \ln|\cos x| + C$

Extra Credit. (*3 pts*) First, explain in terms of familiar Maclaurin series, why $\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

Second, how accurate is $S_9 = \sum_{n=0}^{9} \frac{(-1)^n}{n!}$ as an approximation to $1/e$? Write your bound on the error in the box below.

Note $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Substitute $x=-1$ to get $\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. This series is alternating and converges so $|R_N| \le b_{N+1}$. Here $N=9$, so $|R_9| \le b_{10} = \frac{1}{10!}$.

$$|R_9| = \left| S_9 - \frac{1}{e} \right| \le \boxed{\frac{1}{10!}}$$

You may find the following **trigonometric formulas** useful. However, there are other trig. formulas, not listed here, which you should have in memory, or which can be derived from these.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$
$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$
$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

EXTRA SPACE FOR ANSWERS

Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n \to \infty} a_n$.	If $\lim_{n \to \infty} a_n = 0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n \to \infty} a_n \neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\lvert r \rvert < 1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
	If $\lvert r \rvert \geq 1$, the series diverges.	
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
	If $p \leq 1$, the series diverges.	
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n \to \infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.

Series or Test	Conclusions	Comments
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx.$	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n \to \infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert.$	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n \to \infty} \sqrt[n]{\lvert a_n \rvert}.$	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series where $\lvert a_n \rvert = b_n^n.$
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	