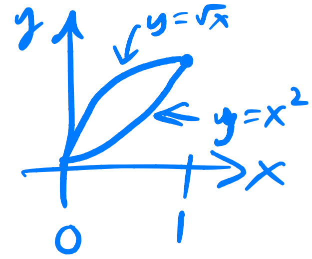
Name: SOLUTIONS

Math 252 Calculus 2 (Bueler) Thursday, 6 October 2022

Midterm Exam 1

No book, notes, electronics, calculator, or internet access. 100 points possible. 70 minutes maximum.

1. (*7 pts*) Compute the area between the curves $y = x^2$ and $y = \sqrt{x}$ on the interval $0 \le x \le 1$. (*Hint. Be careful about which curve is above the other.*)



$$A = \int_0^1 \sqrt{x} - x^2 \, dx$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

2. (*6 pts*) Completely set up, but do not evaluate, a definite integral for the **length** of the curve $y = \sqrt{x}$ on the interval $x = 1$ to $x = 4$.

$$f(x) = x^{1/2}$$

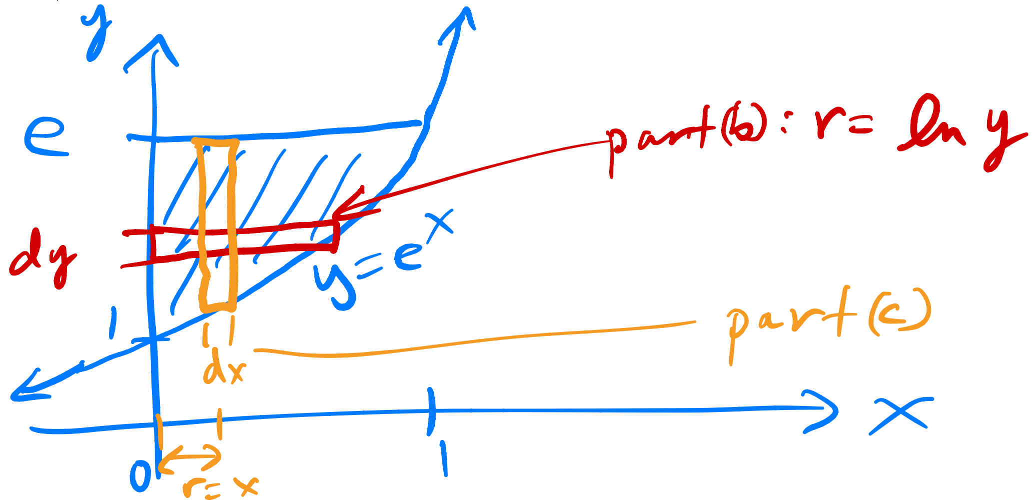
$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$L = \int_1^4 \sqrt{1 + f'(x)^2} \, dx$$

$$= \int_1^4 \sqrt{1 + \frac{1}{4x}} \, dx$$

y axis

3. **(a)** (*4 pts*) Sketch the region bounded by the curves $y = e^x$, $x = 0$ and $y = e$. (*Hint. Double-check this part!*)



(b) (*4 pts*) Use the **slicing (disks/washers)** method to completely set up, but not evaluate, a definite integral for the volume of the solid of revolution formed by rotating the region in part **(a)** around **the** y**-axis**.

$$V = \int_1^e \pi (\ln y)^2 \, dy \qquad \text{(discs)}$$

(c) (*4 pts*) Use the **shells** method to completely set up, but not evaluate, a definite integral for the volume of the same solid of revolution as in part **(b)**.

height

$$V = \int_0^1 2\pi x \cdot \underbrace{(e - e^x)}_{\text{height}} \cdot dx \qquad \text{(shells)}$$

(d) (*4 pts*) Evaluate one of the integrals in parts **(b)** or **(c)** to find the volume.

from (c):

$$V = 2\pi e \int_0^1 x \, dx - 2\pi \int_0^1 x e^x \, dx$$

$$= 2\pi e \left[\frac{x^2}{2}\right]_0^1 - 2\pi \left(\left[x e^x\right]_0^1 - \int_0^1 e^x \, dx \right)$$

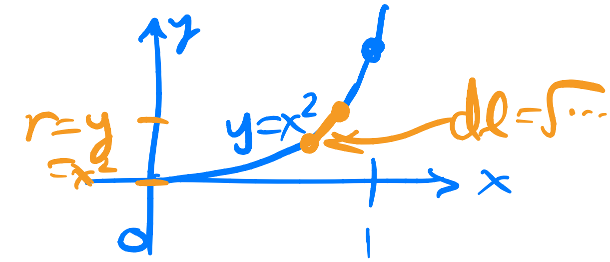
u=x ... du=e^x dx

$$= 2\pi e \cdot \frac{1}{2} - 2\pi\left(e - \left[e^x\right]_0^1\right) = \pi e - 2\pi(e - e + 1)$$

$$= \boxed{\pi(e-2)}$$

4. (*6 pts*) Completely set up, but do not evaluate, a definite integral for the **surface area** of the surface created when the curve $y = x^2$ on the interval $x = 0$ to $x = 1$ is rotated around **the x-axis**.

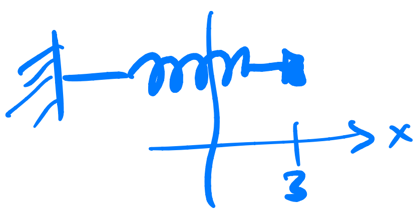
$$A = \int_0^1 2\pi x^2 \sqrt{1 + (2x)^2}\, dx$$



$$= 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2}\, dx$$

5. It takes a force of 4 Newtons to hold a spring 3 centimeters from its equilibrium.

(a) (*3 pts*) What is the spring constant k in Hooke's Law (i.e. $F = kx$)?



$$4 = k \cdot 3$$

$$k = \frac{4}{3}\ \frac{N}{cm} = \frac{400}{3}\ \frac{N}{m} \qquad (\leftarrow \text{either})$$

(b) (*6 pts*) How much **work** is done to compress the spring 6 centimeters from its equilibrium? Simplify your answer and include units.

$$W = \int_0^6 F(x)\,dx = \int_0^6 kx\,dx$$

$$= k \cdot \frac{x^2}{2}\Big]_0^6 = \frac{4}{3} \cdot \frac{6^2}{2} = 24\ N \cdot cm$$

or:

$$W = \int_0^{0.06} F(x)\,dx = k\ \frac{x^2}{2}\Big]_0^{0.06}$$

$$= \frac{400}{3} \cdot \frac{(6 \times 10^{-2})^2}{2} = 4 \cdot 6 \cdot 10^2 \cdot 10^{-4}$$

$$= 24 \cdot 10^{-2} = 0.24\ Nm = 0.24 J$$

6. Evaluate and simplify the following indefinite and definite integrals.

(a) (*6 pts*) $\int_0^2 5^x \, dx =$ $\frac{1}{\ln 5} 5^x \Big]_0^2 = \frac{5^2 - 1}{\ln 5}$

$= \frac{24}{\ln 5}$

(b) (*6 pts*) $\int \cot\theta \, d\theta =$ $\int \frac{\cos\theta}{\sin\theta} d\theta = \int \frac{du}{u}$

$[u = \sin\theta]$

$= \ln|u| + c = \ln|\sin\theta| + c$

(c) (*6 pts*) $\int \cos(7t)\sin(7t)\, dt =$ $\int u \frac{du}{7} = \frac{1}{7}\frac{u^2}{2} + c$

$u = \sin(7t)$

$du = \cos(7t) \cdot 7 \, dt$

$\frac{du}{7} = \cos(7t) dt$

$= \frac{\sin^2(7t)}{14} + C$

(d) (*6 pts*) $\int_0^{\pi/2} \sin^3 x \, dx = \int_0^{\pi/2} \sin^2 x \cdot \sin x \, dx$

$$= \int_0^{\pi/2} (1-\cos^2 x) \sin x \, dx$$

$u = \cos x$
$du = -\sin x dx$
$-du = \sin x dx$

$$= \int_1^0 (1-u^2)(-du)$$

$$= \int_0^1 (1-u^2)\,du = u - \frac{u^3}{3}\Big]_0^1$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

(e) (*6 pts*) $\int x^2 \sin x \, dx = x^2(-\cos x) - \int (-\cos x) \cdot 2x \, dx$

$u = x^2$ $\quad v = -\cos x$
$du = 2x\,dx$ $\quad dv = \sin x\,dx$

$$= -x^2 \cos x + 2 \int x \cos x \, dx$$

$u = x$ $\quad v = \sin x$
$du = dx$ $\quad dv = \cos x\,dx$

$$= -x^2 \cos x + 2\left(x \sin x - \int \sin x \, dx\right)$$

$$= -x^2 \cos x + 2x \sin x - 2(-\cos x) + C$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C$$

(f) (*6 pts*) $\int \sec x \, dx =$ $\int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$ } the trick

$$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \, dx$$

$$\left[\begin{array}{l} u = \sec x + \tan x \\ du = (\sec x \tan x + \sec^2 x) dx \end{array} \right]$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C = \ln|\sec x + \tan x| + C$$

(g) (*6 pts*) $\int \sin(7x)\cos(3x) \, dx =$ $\int \frac{1}{2} \sin((7-3)x) + \frac{1}{2} \sin((7+3)x) \, dx$

[from last page]

$$= \frac{1}{2} \int \sin(4x) + \sin(10x) \, dx = \frac{1}{2}\left(-\frac{\cos(4x)}{4} - \frac{\cos(10x)}{10}\right) + C$$

$$= -\frac{1}{8}\cos(4x) - \frac{1}{20}\cos(10x) + C$$

7. (*8 pts*) Evaluate and simplify the indefinite integral:

$$\int \frac{x^2+x+1}{x^3+x}\,dx = \int \frac{x^2+x+1}{x(x^2+1)}\,dx$$

$$\frac{x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x^2+x+1 = A(x^2+1) + (Bx+C)x$$

$$= (A+B)x^2 + Cx + A$$

$A=1$, $C=1$, $A+B=1$

$\therefore B=0$

$$= \int \frac{1}{x} + \frac{1}{x^2+1}\,dx$$

$$= \ln|x| + \arctan x + C$$

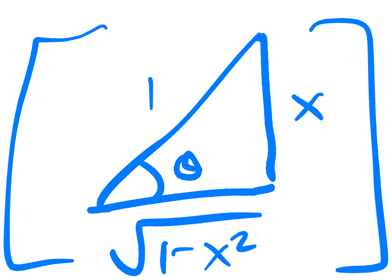
8. (*8 pts*) Evaluate and *fully* simplify the indefinite integral.
(*Hint.* $(\tan\theta)' = \sec^2\theta$ and $(\cot\theta)' = -\csc^2\theta$.)

$$\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = \int \frac{1}{\sin^2\theta\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$[x = \sin\theta]$

$$= \int \frac{\cancel{\cos\theta}}{\sin^2\theta\cdot\cancel{\cos\theta}}\,d\theta = \int \csc^2\theta\,d\theta$$

$$= -\cot\theta + c = -\frac{\sqrt{1-x^2}}{x} + C$$



Extra Credit. (*3 pts*) Compute and simplify the integral

$\underbrace{\int \sec^3\theta \, d\theta}_{=I} =$ $\int \sec\theta \cdot \sec^2\theta \, d\theta$

$\leftarrow \begin{bmatrix} u = \sec\theta & v = \tan\theta \\ du = \sec\theta\tan\theta\, d\theta & dv = \sec^2\theta\, d\theta \end{bmatrix}$

$= \sec\theta\tan\theta - \int \tan\theta \sec\theta \tan\theta \, d\theta$

$= \sec\theta\tan\theta - \int \tan^2\theta \sec\theta \, d\theta$

$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta \, d\theta$

$= \sec\theta\tan\theta + \int \sec\theta \, d\theta - \int \sec^3\theta \, d\theta$ $=I$

$= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int \sec^3\theta \, d\theta$

So: $\int \sec^3\theta \, d\theta = \boxed{\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C}$

You may find the following **trigonometric formulas** useful. However, there are other trig. formulas, not listed here, which you should have in memory, or which you know how to derive from these.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$
$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$
$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$