VERSION 2
28 Nov.

Name: SOLUTIONS

Math 252 Calculus 2 (Bueler) 17 November 2022

Midterm Exam 2

No book, electronics, calculator, or internet access. Only "Summary of Convergence Tests" notes allowed. 100 points possible. 70 minutes.

1. (*5 pts*) Verify that $y = e^{2x^2}$ is a solution to the differential equation $y' - 4xy = 0$.

$$y' = e^{2x^2} \cdot 4x$$

$$y' - 4xy = e^{2x^2} \cdot 4x - 4x \cdot e^{2x^2} = 0 \checkmark$$

2. Compute and simplify the improper integrals, or show they diverge. Use correct limit notation.

(a) (*5 pts*) $\displaystyle\int_0^1 \frac{dx}{x^3} = \lim_{t\to 0^+} \int_t^1 x^{-3}\,dx = \lim_{t\to\infty} \left[\frac{-x^{-2}}{2}\right]_t^1$

$$= \lim_{t\to 0^+} \left[-\frac{1}{2} + \frac{1}{2t^2}\right] = +\infty \quad \text{so diverges}$$

(b) (*5 pts*) $\displaystyle\int_1^\infty 2xe^{-x^2}\,dx = \lim_{t\to\infty} \int_1^t 2xe^{-x^2}\,dx$ $\left[\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}\right]$

$$= \lim_{t\to\infty} \int_1^{t^2} e^{-u}\,du = \lim_{t\to\infty} \left[-e^{-u}\right]_1^{t^2}$$

$$= \lim_{t\to\infty} \left[-e^{-t^2} + e^{-1}\right] = 0 + e^{-1} = \frac{1}{e}$$

3. Do the following series converge or diverge? Show your work, including naming any test you use.

(a) (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2}$

$\frac{n+1}{n^2} \geq \frac{n}{n^2} = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic)

so diverges by comparison test

(or limit comparison test, or integral test)

(b) (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{5n-1}$

$\lim_{n\to\infty} \frac{2n+1}{5n-1} \overset{L'H}{=} \frac{2}{5} \neq 0$

diverges by divergence test

(c) (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2n}}$

$b_n = \frac{1}{\sqrt{2n}}$. $\lim_{n\to\infty} b_n \overset{\frac{1}{\infty}}{=} 0$

b_n decreases

so converges by AST

(d) *(5 pts)* $\displaystyle\sum_{n=0}^{\infty} \frac{2^n n!}{(n+2)!}$ $= \displaystyle\sum_{n=0}^{\infty} \frac{2^n \cancel{n!}}{(n+2)(n+1)\cancel{n!}} = \sum_{n=0}^{\infty} \frac{2^n}{(n+2)(n+1)}$

$$\lim_{n\to\infty} \frac{2^n}{(n+2)(n+1)} \underset{\times 2}{\overset{L'H}{=}} \lim_{n\to\infty} \frac{(\ln 2)^2\, 2^n}{2} = +\infty$$

diverges by divergence test

(or ratio test : $\rho = \cdots = 2 > 1$ $\therefore$ diverges)

(e) *(5 pts)* $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n+3}{2n-1}\right)^n$

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+3}{2n-1}\right)^n} = \lim_{n\to\infty} \frac{n+3}{2n-1}$$

$$\overset{L'H}{=} \frac{1}{2} < 1$$

converges by root test

4. (*5 pts*) Does the following series converge or diverge? Show your work, including naming any test you use. (*Hint.* Previous problem? Or another test?)

$$\sum_{n=1}^{\infty} \frac{2n}{e^{(n^2)}}$$

integral test using 2(b):

$$\int_1^{\infty} 2x e^{-x^2} dx = \frac{1}{e} \text{ converges}$$

so series converges

(or ratio test or root test)

5. (*5 pts*) Compute and simplify the value of the infinite series $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^{n+1}$.

geometric series

$$= \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \cdots$$

so $a = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$, $r = \frac{2}{5} < 1$ (converges)

So: $$\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^{n+1} = \frac{4/25}{1 - 2/5} = \frac{\frac{4}{25}}{\frac{3}{5}} = \frac{4}{\cancel{25}\,5} \cdot \frac{\cancel{5}}{3}$$

$$= \frac{4}{15}$$

6. Consider the infinite series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots$

(a) (*5 pts*) Write the series using sigma ($\sum$) notation.

$$= \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}}_{} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}}_{}$$

either is fine

(b) (*5 pts*) Compute and simplify S_3, the partial sum of the first three terms.

$$S_3 = 1 - \frac{1}{4} + \frac{1}{9} = \frac{36-9+4}{36} = \frac{31}{36}$$

(c) (*5 pts*) Does this series converge absolutely, conditionally, or neither (diverge)? Show your work, identify any test(s) used, and circle one answer.

$$\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ $p=2$ series converges

CONVERGES ABSOLUTELY

CONVERGES CONDITIONALLY

DIVERGES

7. Use the well known geometric series $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ to find power series representations for the following functions. Show your work.

(a) (*5 pts*) $\frac{1}{1+x^3}$

$r = -x^3$

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

(b) (*7 pts*) $\ln(1+x)$

CORRECTED

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$x=0$: $0 = \ln 1 = C + 0 \therefore C = 0$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

8. (*7 pts*) If $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$, find a power series representation for $f'(5x)$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{\sqrt{n}} = \sum_{n=1}^{\infty} \sqrt{n}\, x^{n-1}$$

either is fine

$$f'(5x) = \sum_{n=1}^{\infty} \sqrt{n}\,(5x)^{n-1} = \sum_{m=0}^{\infty} \sqrt{m+1}\,(5x)^m$$

9. Find the interval of convergence of the following power series.

(a) *(8 pts)* $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n!}$

ratio test

$$\rho = \lim_{n\to\infty} \frac{\frac{2^{n+1}|x|^{n+1}}{(n+1)!}}{\frac{2^n |x|^n}{n!}} = \lim_{n\to\infty} \frac{2^{\cancel{n+1}} |x|^{\cancel{n+1}} \cancel{n!}}{(n+1)\cancel{n!}\, \cancel{2^n} \cancel{|x|^n}}$$

$$= \lim_{n\to\infty} \frac{2|x|}{(n+1)} = 0 < 1$$

$\therefore$ $I = (-\infty, \infty)$ is int. of conv.

(b) *(8 pts)* $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n\, 3^n}$

ratio test

$$\rho = \lim_{n\to\infty} \frac{\frac{|x-1|^{n+1}}{(n+1)3^{n+1}}}{\frac{|x-1|^n}{n3^n}} = \lim_{n\to\infty} \frac{|x-1|^{\cancel{n+1}}\, n\cancel{3^n}}{(n+1)\cancel{3^n}\, 3\, \cancel{|x-1|^n}}$$

(or root test)

$$= \frac{|x-1|}{3} \lim_{n\to\infty} \frac{n}{n+1} \overset{L'H}{=} \frac{|x-1|}{3} \cdot 1 = \frac{|x-1|}{3} < 1$$

$\iff -3 < x-1 < 3 \iff -2 < x < 4$

$\underline{x=-2:}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges AST

$\underline{x=4:}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges harmonic

$I = [-2, 4)$ is int. of conv.

Extra Credit. (*3 pts*) The function $f(x) = \arctan(x)$ can be represented by the power series

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Suppose I choose $x = 1/\sqrt{3}$ and compute the partial sum $S_{20} = \sum_{n=0}^{20} \frac{(-1)^n (1/\sqrt{3})^{2n+1}}{2n+1}$ as an approximation to $\arctan(1/\sqrt{3}) = \frac{\pi}{6}$.

How accurate is this approximation? Use a known fact about remainders of alternating series.

$$\frac{\pi}{6} = \arctan\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (1/\sqrt{3})^{2n+1}}{2n+1} = S$$

$$|R_{20}| = |S - S_{20}| \leq b_{21} = \frac{(1/\sqrt{3})^{43}}{43} = \frac{1}{43 \cdot 3^{43/2}}$$

$$= \frac{1}{43(3^{21.5})}$$

very small!

$$\left|S_{20} - \frac{\pi}{6}\right| \leq \boxed{\frac{1}{43 \cdot 3^{21.5}}}$$

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