Math F252 Final Exam Fall 2023

Name: Solutions

Rules:

You have 2 hours to complete this midterm.

Partial credit will be awarded, but you must show your work.

You may have a single sheet of paper written on the front only.

Calculators and books are not allowed.

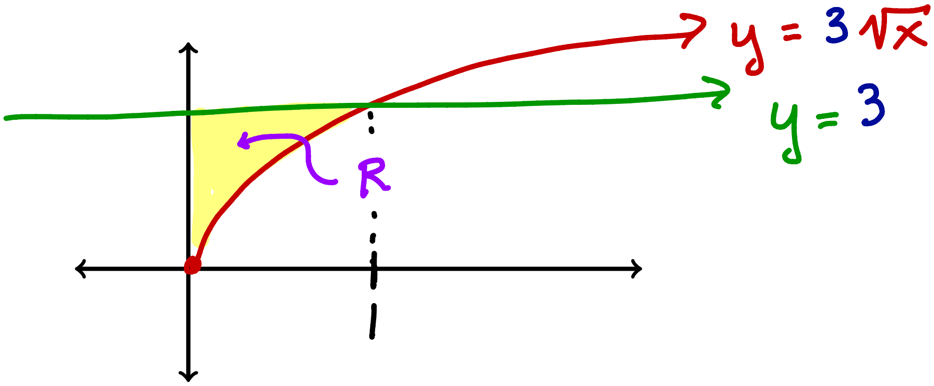
Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	5	
3	5	
4	10	
5	10	
6	10	
7	8	
8	8	
9	6	
10	12	
11	8	
12	8	
Extra Credit	5	
Total	100	

1. (10 points) Let R be the region of the plane bounded by $y = 3\sqrt{x}$, the y-axis, and $y = 3$.

   (a) Sketch the region R. Label at least three points on your graph.



   (b) Find the area of the region R. Your final answer should be simplified.

$$A = \int_0^1 (3 - 3x^{\frac{1}{2}})\,dx = 3x - 2x^{\frac{3}{2}}\Big]_0^1$$

$$= (3-2) - 0 = 1$$

   (c) Find the volume of the solid obtained by rotating the region R about the x-axis. Your final answer should be simplified.

$$V = \pi \int_0^1 \left(3^2 - (3\sqrt{x})^2\right)dx = \pi \int_0^1 (9 - 9x)\,dx$$

$$= \pi\left(9x - \frac{9}{2}x\right]_0^1 = \pi\left(9 - \frac{9}{2}\right) = \frac{9\pi}{2}$$

2. (5 points) A 1-meter long rod oriented along the x-axis on the interval [0, 1] has density $\rho(x) = xe^{3x}$ grams per meter at position x meters. Find the mass of the rod. Include units in your answer.

$$\text{mass} = \int_0^1 xe^{3x}\,dx = \frac{1}{3}xe^{3x}\Big]_0^1 - \frac{1}{3}\int_0^1 e^{3x}\,dx$$

$$\left[\begin{array}{ll} u = x & dv = e^{3x}dx \\ du = dx & v = \frac{1}{3}e^{3x} \end{array}\right] = \frac{1}{3}e^3 - \frac{1}{9}\left[e^{3x}\right]_0^1 = \frac{1}{3}e^3 - \frac{1}{9}(e^3 - 1)$$

$$= \frac{1}{3}e^3 - \frac{1}{9}e^3 + \frac{1}{9} = \frac{1}{9} + \frac{2}{9}e^3$$

3. (5 points) Evaluate the definite integral: $\int_0^1 \frac{5x+1}{(x+1)(2x+1)}\,dx = \int_0^1 \left(\frac{4}{x+1} - \frac{3}{2x+1}\right)dx$

partial fractions

$$\frac{5x+1}{(x+1)(2x+1)} = \frac{A}{x+1} + \frac{B}{2x+1}$$

$$5x+1 = A(2x+1) + B(x+1)$$

$x=-1$: $-4 = -A$. $\boxed{A=4}$

$x = -\frac{1}{2}$: $-\frac{3}{2} = \frac{1}{2}B$

$$\boxed{B=-3}$$

$$= 4\ln|x+1| - \frac{3}{2}\ln|2x+1|\Big]_0^1$$

$$= 4\ln(2) - \frac{3}{2}\ln(3)$$

4. (10 points) Evaluate the following indefinite integrals.

(a) $\int \sec^4(\theta)\, d\theta$ $= \int \sec^2\theta \sec^2\theta \, d\theta$

$= \int (1+\tan^2\theta)(\sec^2\theta \, d\theta) = \int (1+u^2)\,du = u + \frac{1}{3}u^3 + C$

let $u = \tan\theta$
$du = \sec^2\theta \, d\theta$

$= \tan\theta + \frac{1}{3}\tan^3\theta + C$

(b) $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ $= \int \frac{4\sin^2\theta \cdot 2\cos\theta \, d\theta}{2\cos\theta} = \int 4\sin^2\theta \, d\theta$

Let $x = 2\sin\theta$

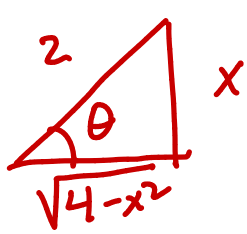
$dx = 2\cos\theta \, d\theta$

$4 - x^2 = 4 - 4\sin^2\theta$

$= 4\cos^2\theta$

$\frac{x}{2} = \sin\theta$

$\theta = \arcsin\left(\frac{x}{2}\right)$



$= \int 2(1-\cos(2\theta))\,d\theta$

$= 2\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C$

$= 2\theta - \sin(2\theta) + C$

$= 2\theta - 2\sin(\theta)\cos\theta + C$

$= 2\arcsin\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$

$= 2\arcsin\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C$

5. (10 points) Determine whether the series is convergent or divergent. Note that to earn full credit, you work must include the name of the test being applied, a clear application of the test, and a conclusion.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+1}{n^3+1}$

L.C.T. Compare to divergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{n^2+1}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+1}{n^3+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3+n}{n^3+1} = 1$$

So the series diverges.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$

A.S.T : $b_n = \frac{1}{\sqrt{2n+1}}$, $b_{n+1} = \frac{1}{\sqrt{2n+3}} \leq \frac{1}{\sqrt{2n+1}} = b_n$

So b_n's are decreasing

$$\lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0.$$

So it converges.

6. (10 points) Determine whether the series is convergent or divergent. Note that to earn full credit, you work must include the name of the test being applied, a clear application of the test, and a conclusion.

   (a) $\displaystyle\sum_{n=0}^{\infty} \ln\left(\frac{2n}{3n+5}\right)$

Divergence test

$$\lim_{n\to\infty} \ln\left(\frac{2n}{3n+5}\right) = \ln\left(\lim_{n\to\infty}\left(\frac{2n}{3n+5}\right)\right) = \ln\left(\frac{2}{3}\right) \neq 0.$$

So it diverges.

   (b) $1 + e + \dfrac{e^2}{2!} + \dfrac{e^3}{3!} + \dfrac{e^4}{4!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$

• Fast soln: $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$; So $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!} = e^e$. So it converges.

• Alt: Use ratio test.

$$\lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \lim_{n\to\infty} \frac{e}{n+1} = 0.$$

So it converges.

7. (8 points) Evaluate the improper integral $\int_2^\infty \frac{dx}{x(\ln(x))^2}$ or demonstrate that it is divergent. Use correct limit notation.

$$\lim_{n\to\infty} \int_2^n \frac{(\ln x)^{-2}}{x}\,dx = \lim_{n\to\infty} -(\ln x)^{-1}\Big]_2^n$$

$$= \lim_{n\to\infty} -\frac{1}{\ln(n)} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

8. (8 points) Find the interval of convergence for the series $\sum_{n=1}^\infty \frac{(x-2)^n}{n3^n}$

root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{n\,3^n}\right|} = \lim_{n\to\infty} \frac{|x-2|}{\sqrt[n]{n}\,3} = \frac{|x-2|}{3} < 1$$

So $|x-2|<3$ or $-3 < x-2 < 3$ or $-1 < x < 5$.

End pts:

$x=-1$: $\sum \frac{(-1)^n}{n}$ alt. harm

answer $[-1,5)$

$x=5$: $\sum \frac{1}{n}$ harmonic

9. (6 points) Find the Taylor series for $f(x) = e^{5x}$ centered at $a = -1$.

$f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 2^2 e^{2x}$

$\vdots$

$f^{(k)}(x) = 2^k e^{2x}$

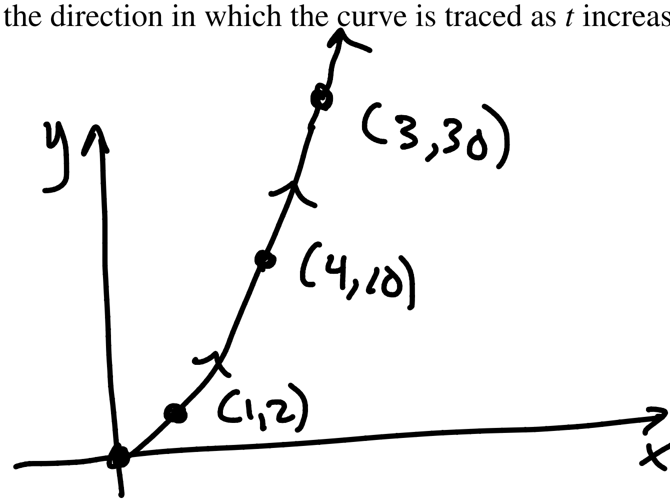
$f^{(k)}(-1) = 2^k e^{-2}$

$$f(x) = e^{2x} = \sum_{n=0}^{\infty} \frac{2^n e^{-2} (x+1)^n}{n!}$$

10. (12 points) Answer the questions about the parametric equations $x(t) = t^2$ and $y(t) = t^3 + t$, where $0 \leq t$.

(a) Make a rough sketch of the curve defined by the parametric equations. Plot at least 4 points and indicate with arrows the direction in which the curve is traced as t increases.

t	x	y
0	0	0
1	1	2
2	4	10
3	9	30



(b) Write an equation of the line tangent to the curve when $t = 1$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2+1}{2t}$$

$$\left.\frac{dy}{dt}\right|_{t=1} = \frac{4}{2} = 2$$

$$y - 2 = 2(x-1)$$

or

$$y = 2x$$

(c) Use the second derivative to demonstrate that the curve is concave up when $t = 1$.

$$\frac{dy}{dx} = \frac{3}{2}t + \frac{1}{2}t^{-1}$$

$$\frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{3}{2} - \frac{1}{2}t^{-2}$$

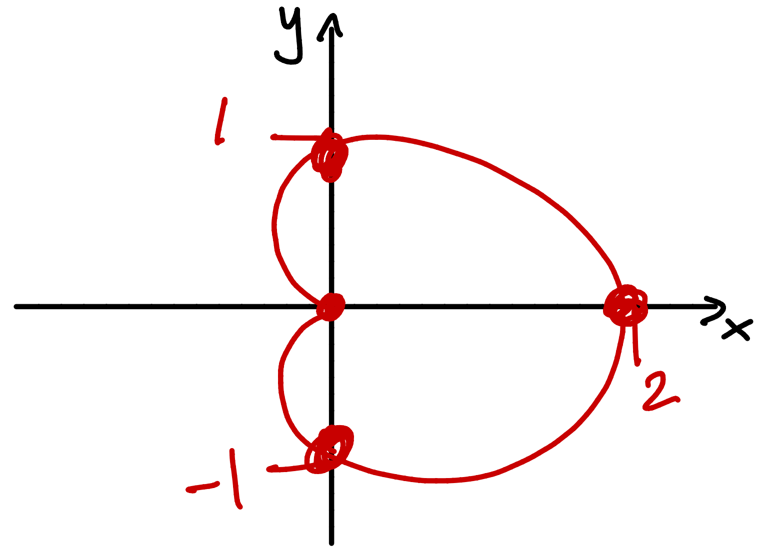
$$\frac{d^2y}{dx^2} = \frac{3/2 - \frac{1}{2}t^{-2}}{2t}$$

at $t=1$

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2} - \frac{1}{2}}{2} > 0$$

11. (8 points) Make a careful and reasonably large sketch of the polar curve $r = 1 + \cos(\theta)$. To earn full credit, you must label at least 4 points on your graph.

θ	r
0	2
$\pi/2$	1
π	0
$3\pi/2$	1
2π	2



12. (8 points) Find the area inside the polar curve in the previous problem.

$$A = \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\, d\theta = \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{\pi} 1 + 2\cos\theta + \frac{1}{2}(1+\cos(2\theta))\, d\theta = \int_0^{\pi} \left[\frac{3}{2} + 2\cos(\theta) + \frac{1}{2}\cos(2\theta)\right] d\theta$$

$$= \frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{4}\sin(2\theta)\Big]_0^{\pi} = \left(\frac{3\pi}{2} + 0 + 0\right) - (0+0+0)$$

$$= \frac{3\pi}{2}$$

Extra Credit (5 points) Determine all the values of p for which the series $\sum_{n=0}^{\infty} \frac{4^{pn}}{3^n}$ converges or explain why it is not possible for the series to converge for any value of p.

$\sum_{n=0}^{\infty} \left(\frac{4^p}{3}\right)^n$ is geometric with $r = \frac{4^p}{3}$.

We need $\left|\frac{4^p}{3}\right| < 1$ for convergence.

So $4^p < 3$ or $p < \frac{\ln(3)}{\ln(4)}$