

Name: Solutions

Rules:

You have 2 hours to complete this midterm.

Partial credit will be awarded, but you must show your work.

You may have a single sheet of paper written on the front only.

Calculators and books are not allowed.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	5	
4	5	
5	10	
6	10	
7	8	
8	8	
9	6	
10	12	
11	8	
12	8	
Extra Credit	5	
Total	100	

1. (10 points) Evaluate the following indefinite integrals.

$$(a) \int \sec^4(\theta) d\theta = \int \sec^2 \theta \sec^2 \theta d\theta$$

$$= \int (1 + \tan^2 \theta) (\sec^2 \theta d\theta) = \int (1 + u^2) du = u + \frac{1}{3} u^3 + C$$

$$\text{let } u = \tan \theta$$

$$du = \sec^2 \theta d\theta$$

$$= \tan \theta + \frac{1}{3} \tan^3 \theta + C$$

$$(b) \int \frac{x^2}{\sqrt{4-x^2}} dx = \int \frac{4 \sin^2 \theta \cdot 2 \cos \theta d\theta}{2 \cos \theta} = \int 4 \sin^2 \theta d\theta$$

$$\text{let } x = 2 \sin \theta$$

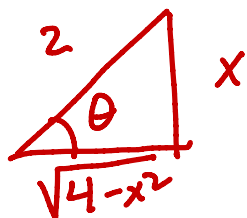
$$dx = 2 \cos \theta d\theta$$

$$4 - x^2 = 4 - 4 \sin^2 \theta$$

$$= 4 \cos^2 \theta$$

$$\frac{x}{2} = \sin \theta$$

$$\theta = \arcsin\left(\frac{x}{2}\right)$$



$$= \int 2(1 - \cos(2\theta)) d\theta$$

$$= 2\left(\theta - \frac{1}{2} \sin(2\theta)\right) + C$$

$$= 2\theta - \sin(2\theta) + C$$

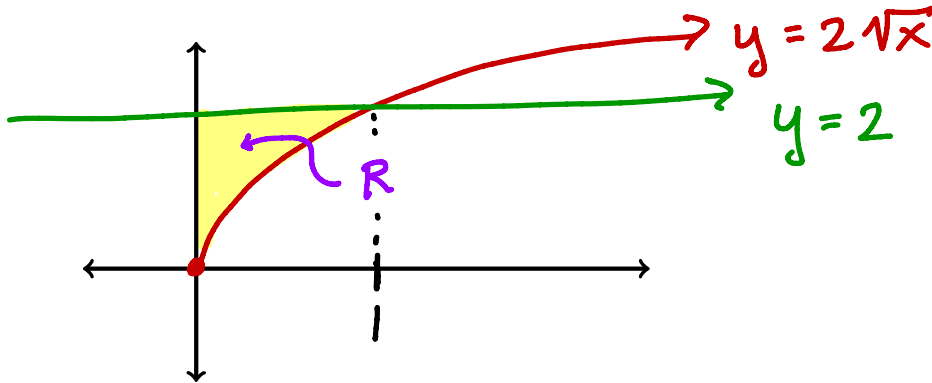
$$= 2\theta - 2 \sin(\theta) \cos \theta + C$$

$$= 2 \arcsin\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right) \left(\frac{\sqrt{4-x^2}}{2}\right) + C$$

$$= 2 \arcsin\left(\frac{x}{2}\right) - \frac{x \sqrt{4-x^2}}{2} + C$$

2. (10 points) Let R be the region of the plane bounded by $y = 2\sqrt{x}$, the y -axis, and $y = 2$.

(a) Sketch the region R . Label at least three points on your graph.



(b) Find the area of the region R . Your final answer should be simplified.

$$A = \int_0^1 (2 - 2x^{\frac{1}{2}}) dx = 2x - \frac{4}{3} x^{\frac{3}{2}} \Big|_0^1$$

$$= 2 - \frac{4}{3} = \frac{2}{3}$$

(c) Find the volume of the solid obtained by rotating the region R about the x -axis. Your final answer should be simplified.

$$V = \pi \int_0^1 (2^2 - (2\sqrt{x})^2) dx = \pi \int_0^1 (4 - 4x) dx$$

$$= \pi (4x - 2x^2) \Big|_0^1 = 2\pi$$

3. (5 points) Evaluate the definite integral: $\int_0^1 \frac{5x+1}{(x+1)(2x+1)} dx = \int_0^1 \left(\frac{4}{x+1} - \frac{3}{2x+1} \right) dx$

partial fractions

$$\frac{5x+1}{(x+1)(2x+1)} = \frac{A}{x+1} + \frac{B}{2x+1}$$

$$= 4 \ln|x+1| - \frac{3}{2} \ln|2x+1| \Big|_0^1$$

$$5x+1 = A(2x+1) + B(x+1)$$

$$x=-1: -4 = -A. \quad \boxed{A=4}$$

$$= 4 \ln(2) - \frac{3}{2} \ln(3)$$

$$x = -\frac{1}{2}: -\frac{3}{2} = \frac{1}{2} B$$

$$\boxed{B = -3}$$

4. (5 points) A 1-meter long rod oriented along the x-axis on the interval $[0, 1]$ has density $\rho(x) = xe^{3x}$ grams per meter at position x meters. Find the mass of the rod. Include units in your answer.

$$\text{mass} = \int_0^1 x e^{3x} dx = \left[\frac{1}{3} x e^{3x} \right]_0^1 - \frac{1}{3} \int_0^1 e^{3x} dx$$

$$\left. \begin{array}{l} u=x \quad dv=e^{3x} dx \\ du=dx \quad v=\frac{1}{3} e^{3x} \end{array} \right\} = \frac{1}{3} e^3 - \frac{1}{9} \left[e^{3x} \right]_0^1 = \frac{1}{3} e^3 - \frac{1}{9} (e^3 - 1)$$

$$= \frac{1}{3} e^3 - \frac{1}{9} e^3 + \frac{1}{9} = \frac{1}{9} + \frac{2}{9} e^3$$

5. (10 points) Determine whether the series is convergent or divergent. Note that to earn full credit, you work must include the name of the test being applied, a clear application of the test, and a conclusion.

$$(a) \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$$

$$\text{A.S.T: } b_n = \frac{1}{\sqrt{2n+1}}, \quad b_{n+1} = \frac{1}{\sqrt{2n+3}} \leq \frac{1}{\sqrt{2n+1}} = b_n$$

So b_n 's are decreasing ✓

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2n+1}} = 0. \quad \text{So it converges.}$$

$$(b) \sum_{n=0}^{\infty} \frac{n^2+1}{n^3+1}$$

L.C.T. Compare to divergent p-series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n \rightarrow \infty} \frac{\frac{n^2+1}{n^3+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2+1}{n^3+1} \cdot \frac{n}{1} = \lim_{n \rightarrow \infty} \frac{n^3+n}{n^3+1} = 1 \quad \checkmark$$

So the series diverges.

6. (10 points) Determine whether the series is convergent or divergent. Note that to earn full credit, you work must include the name of the test being applied, a clear application of the test, and a conclusion.

$$(a) \sum_{n=0}^{\infty} \ln\left(\frac{2n}{3n+5}\right)$$

Divergence test

$$\lim_{n \rightarrow \infty} \ln\left(\frac{2n}{3n+5}\right) = \ln\left(\lim_{n \rightarrow \infty} \left(\frac{2n}{3n+5}\right)\right) = \ln\left(\frac{2}{3}\right) \neq 0.$$

So it diverges.

$$(b) 1 + e + \frac{e^2}{2!} + \frac{e^3}{3!} + \frac{e^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{e^n}{n!}$$

• Fast soln: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; So $\sum_{n=0}^{\infty} \frac{e^n}{n!} = e^e$. So it converges.

• Alt: Use ratio test.

$$\lim_{n \rightarrow \infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \lim_{n \rightarrow \infty} \frac{e}{n+1} = 0. \text{ So it converges.}$$

7. (8 points) Evaluate the improper integral $\int_2^{\infty} \frac{dx}{x(\ln(x))^2}$ or demonstrate that it is divergent. Use correct limit notation.

$$\lim_{n \rightarrow \infty} \int_2^n \frac{(\ln x)^{-2}}{x} dx = \lim_{n \rightarrow \infty} \left. -(\ln x)^{-1} \right|_2^n$$

$$= \lim_{n \rightarrow \infty} -\frac{1}{\ln(n)} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

8. (8 points) Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}$

root test

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(x-2)^n}{n3^n} \right|} = \lim_{n \rightarrow \infty} \frac{|x-2|}{\sqrt[n]{n} \cdot 3} = \frac{|x-2|}{3} < 1$$

So $|x-2| < 3$ or $-3 < x-2 < 3$ or $-1 < x < 5$.

End pts:

$x = -1$: $\sum \frac{(-1)^n}{n}$ alt. harm

answer $[-1, 5)$

$x = 5$: $\sum \frac{1}{n}$ harmonic

9. (6 points) Find the Taylor series for $f(x) = e^{2x}$ centered at $a = -1$.

$$f(x) = e^{2x}$$

$$f'(x) = 2e^{2x}$$

$$f''(x) = 2^2 e^{2x}$$

⋮

$$f^{(k)}(x) = 2^k e^{2x}$$

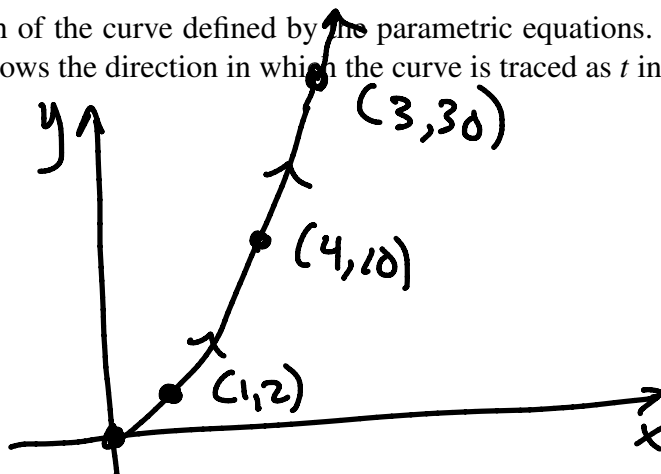
$$f^{(k)}(-1) = 2^k e^{-2}$$

$$f(x) = e^{2x} = \sum_{n=0}^{\infty} \frac{2^n e^{-2} (x+1)^n}{n!}$$

10. (12 points) Answer the questions about the parametric equations $x(t) = t^2$ and $y(t) = t^3 + t$, where $0 \leq t$.

(a) Make a rough sketch of the curve defined by the parametric equations. Plot at least 4 points and indicate with arrows the direction in which the curve is traced as t increases.

t	x	y
0	0	0
1	1	2
2	4	10
3	9	30



(b) Write an equation of the line tangent to the curve when $t = 1$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 1}{2t}$$

$$y - 2 = 2(x - 1)$$

or

$$\left. \frac{dy}{dx} \right|_{t=1} = \frac{4}{2} = 2$$

$$y = 2x$$

(c) Use the second derivative to demonstrate that the curve is ~~always~~ concave up when $t = 1$.

$$\frac{dy}{dx} = \frac{3}{2}t + \frac{1}{2}t^{-1}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2} - \frac{1}{2}t^{-2}}{2t}$$

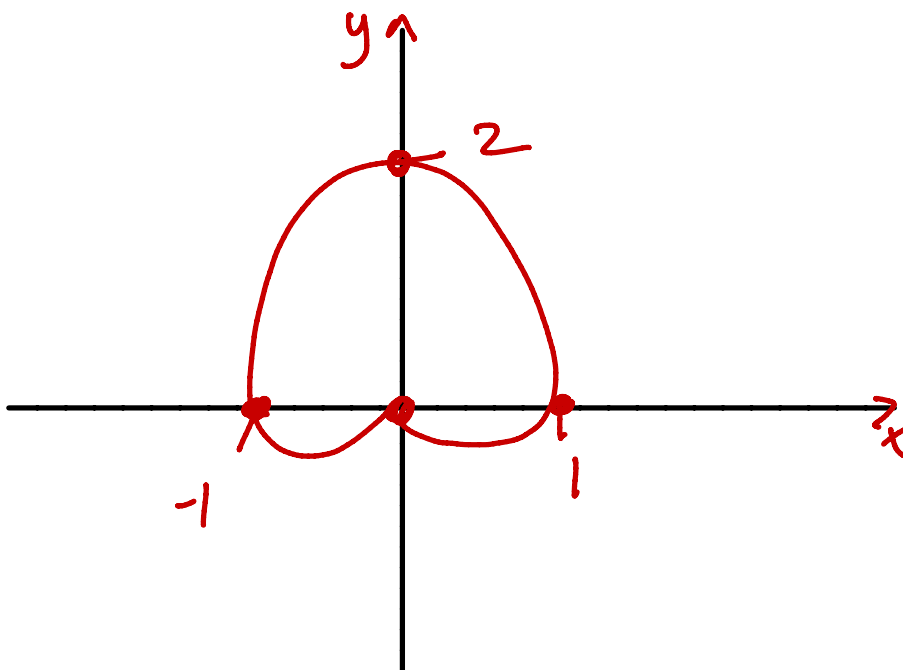
$$\frac{d}{dt} \left[\frac{dy}{dx} \right] = \frac{3}{2} - \frac{1}{2}t^{-2}$$

at $t=1$

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2} - \frac{1}{2}}{2} > 0$$

11. (8 points) Make a careful and reasonably large sketch of the polar curve $r = 1 + \sin(\theta)$. To earn full credit, you must label at least 4 points on your graph.

θ	r
0	1
$\frac{\pi}{2}$	2
π	1
$\frac{3\pi}{2}$	0
2π	1



12. (8 points) Find the area inside the polar curve in the previous problem.

$$\begin{aligned}
 A &= \int_0^{2\pi} \frac{1}{2} (1 + \sin\theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (1 + 2\sin\theta + \sin^2\theta) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left(1 + 2\sin\theta + \frac{1}{2}(1 + \cos 2\theta)\right) d\theta = \frac{1}{2} \int_0^{2\pi} \left(\frac{3}{2} + 2\sin\theta + \frac{1}{2}\cos 2\theta\right) d\theta \\
 &= \frac{1}{2} \left(\frac{3}{2}\theta - 2\cos\theta + \frac{1}{4}\sin(2\theta)\right) \Big|_0^{2\pi} \\
 &= \frac{1}{2} \left((3\pi - 2 + 0) - (0 - 2 + 0)\right) = \frac{3\pi}{2}
 \end{aligned}$$

Extra Credit (5 points) Determine all the values of p for which the series $\sum_{n=0}^{\infty} \frac{4^{pn}}{3^n}$ converges or explain why it is not possible for the series to converge for any value of p .

$$\sum_{n=0}^{\infty} \left(\frac{4^p}{3}\right)^n \text{ is geometric with } r = \frac{4^p}{3}.$$

We need $\left|\frac{4^p}{3}\right| < 1$ for convergence.

$$\text{So } 4^p < 3 \text{ or } p < \frac{\ln(3)}{\ln(4)}$$