Math F252 **Midterm I** **Fall 2023**

Name: Solutions

8:45

Rules:

You have 90 minutes to complete this midterm.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten 3×5 notecard.

Calculators are not allowed.

Place a box around your FINAL ANSWER to each question where appropriate.

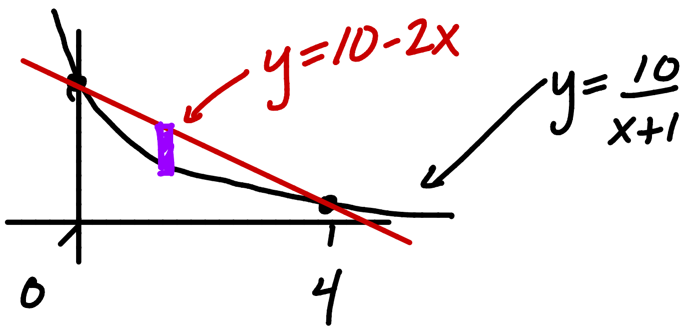
Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	20	
4	12	
5	8	
6	30	
7	10	
Extra Credit	5	
Total	100	

1. (10 points) Find the area between the curves $y = \dfrac{10}{x+1}$ and $y = 10 - 2x$ on the interval $[0, 4]$.

Ans a simplified number.



$$A = \int_0^4 (10-2x) - \left(\frac{10}{x+1}\right) dx = 10x - x^2 - 10\ln(x+1)\Big|_0^4$$

$$= 40 - 16 - 10\ln(5) - (0) = \boxed{24 - 10\ln(5)}$$

2. (10 points) Find the surface area of the volume generated when the curve $y = \frac{x^3}{3}$ from $x = 0$ to $x = 2$ is revolved around the x-axis.

$$y = \tfrac{1}{3}x^3, \quad y' = x^2$$

$$SA = 2\pi \int_0^2 \frac{x^3}{3} \cdot \sqrt{1 + (x^2)^2}\, dx = \frac{2\pi}{3} \int_0^2 x^3 (1 + x^4)^{1/2} dx$$

let $u = 1 + x^4$
$du = 4x^3 dx$
$\frac{1}{4} du = x^3 dx$
if $x = 0$, $u = 1$
$x = 2$, $u = 17$

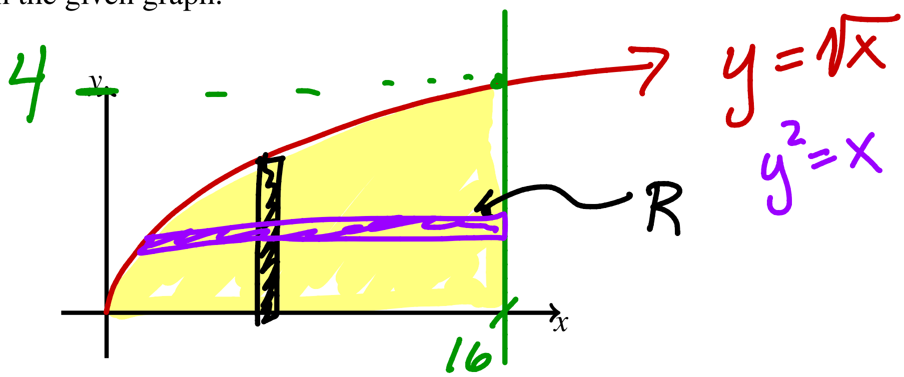
$$= \frac{2\pi}{3} \cdot \frac{1}{4} \int_1^{17} u^{1/2} du = \frac{\pi}{6} \cdot \frac{2}{3} \cdot u^{3/2}\Big|_1^{17}$$

$$= \boxed{\frac{\pi}{9}\left((17)^{3/2} - 1\right)}$$

Change direction

3. (20 points) Let R be the region in the first quadrant bounded by $y = \sqrt{x}$, $y = 0$, and $x = 16$.

   (a) Sketch the region R on the given graph.



   (b) **Use disks/washers** to completely set up, but not evaluate, a definite integral for the volume of the solid of revolution formed by rotating the region in part **(a)** around **the x-axis**.

vertical slice :
$$V = \pi \int_0^{16} (\sqrt{x})^2\,dx = \pi \int_0^{16} x\,dx$$

   (c) **Use shells** to completely set up, but not evaluate, a definite integral for the volume of the same solid of revolution formed by rotating the region in part **(a)** around **the x-axis**.

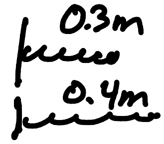
horizontal slice :
$$V = 2\pi \int_0^4 y(16-y^2)\,dy = 2\pi \int_0^4 (16y - y^3)\,dx$$

   (d) **Use disks/washers** to completely set up, but not evaluate, a definite integral for the volume of the solid of revolution formed by rotating the region in part **(a)** around **the y-axis**.

horizontal slice
$$V = \pi \int_0^4 (16)^2 - (y^2)^2\,dy = \pi \int_0^4 (256 - y^4)\,dy$$

   (e) **Use shells** to completely set up, but not evaluate, a definite integral for the volume of the same solid of revolution formed by rotating the region in part **(a)** around **the y-axis**.

vertical axis
$$V = 2\pi \int_0^{16} x \cdot \sqrt{x}\,dx = 2\pi \int_0^{16} x^{3/2}\,dx$$



4. (12 points) A spring has a natural length of 30 cm (or 0.3 m). It takes a force of 40 N to hold the spring at a length of 40 cm (or 0.4 m).

(a) What is the spring constant k in Hooke's Law?

$x = 0.4 - 0.3 = 0.1\,m$
$F = 40N$

$F = kx$
$40 = k \cdot \frac{1}{10}$ so $\boxed{k = 400}$

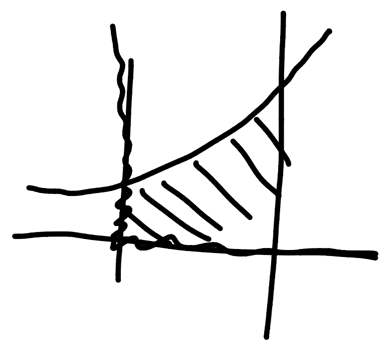
(b) How much **work** is done to stretch the spring to a length of 50 cm (or 0.5 m)? Simplify your answer and include units.

from natural length to..

$$W = \int_0^{0.2} 400x\,dx = 200x^2 \Big|_0^{\frac{2}{10}} = 200\left(\frac{4}{100}\right) = \boxed{8 N \cdot m} \text{ (or J)}$$

mass

5. (8 points) Set up but do not evaluate the three integrals needed to compute the center of mass, $(\bar{x}, \bar{y})$, of the region R bounded by $y = 0$, $x = 0$, $x = 2$, and $f(x) = 2e^x$. Then, fill in the blanks at the bottom to show how to compute the values of $(\bar{x}, \bar{y})$.



$$m = \rho \int_0^2 2e^x\,dx$$

$$M_y = \rho \int_0^2 2xe^x\,dx$$

$$M_x = \rho \int_0^2 \frac{1}{2}\left(2e^x\right)^2 dx = 2\rho \int_0^2 e^{2x}\,dx$$

$$\bar{x} = \frac{\boxed{M_y}}{\boxed{m}} \qquad \bar{y} = \frac{\boxed{M_x}}{\boxed{m}}$$

6. (30 points) Evaluate the following indefinite integrals. Show your work and simplify your answers.

(a) $\int x^2 e^x \, dx$

$$\begin{cases} u = x^2 & dv = e^x\,dx \\ du = 2x\,dx & v = e^x \end{cases}$$

$$= x^2e^x - 2\int x e^x \, dx$$

$$\begin{cases} u = x & dv = e^x\,dx \\ du = dx & v = e^x \end{cases}$$

$$= x^2e^x - 2\left(x e^x - \int e^x \, dx\right)$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

$$= \underline{\underline{e^x(x^2 - 2x + 2) + C}}$$

(b) $\int \sin^3(\theta)\cos^2(\theta)\, d\theta = \int \sin^2\theta \cos^2\theta \cdot \sin\theta \, d\theta$

$$= \int (1 - \cos^2\theta)\cos^2\theta \sin\theta \, d\theta$$

let $u = \cos\theta$
$du = -\sin\theta \, d\theta$

$$= -\int (1-u^2)u^2 \, du = \int (u^4 - u^2)\, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \underline{\underline{\frac{1}{5}(\cos\theta)^5 - \frac{1}{3}(\cos\theta)^3 + C}}$$

(c) $\int \frac{4-x^2}{x^3+2x}\,dx$ $= \int\left(\frac{2}{x} - \frac{3x}{x^2+2}\right)dx$

$x^3+2x = x(x^2+2)$

$\frac{4-x^2}{x^3-2x} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$

$= 2\ln|x| - \frac{3}{2}\ln(x^2+2) + C$

$4-x^2 = A(x^2+2)+(x)(Bx+C)$

$x=0$: $4 = 2A$. So $A=2$

$-x^2 = (A+B)x^2$. So $A+B=-1$

So $B=-3$

$0x = Cx$. So $C=0$

(d) $\int x\sec^2(x)\,dx$

$u = x$ $dv = \sec^2 x\,dx$

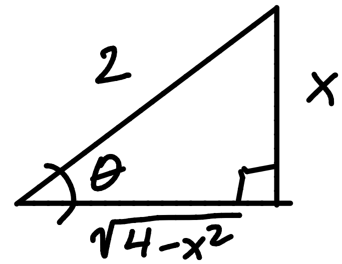
$du = dx$ $v = \tan x$

$= x\tan x - \int \tan x\,dx$

$= x\tan x - \int \frac{\sin x\,dx}{\cos x}$

$= x\tan x + \ln|\cos x| + C$

7. (10 points) Evaluate the integral $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ using trigonometric substitution. You must **fully** simplify your answer.



let $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$

$$\sqrt{4-x^2} = 2\cos\theta$$

$$\int \frac{x^2\,dx}{\sqrt{4-x^2}} = \int \frac{4\sin^2\theta \cdot 2\cos\theta\, d\theta}{2\cos\theta\, d\theta} = 4\int \sin^2\theta\, d\theta$$

$$= 2\int \left(1 - \cos(2\theta)\right) d\theta = 2\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C$$

$$= 2\left(\theta - \sin(\theta)\cos(\theta)\right) + C$$

$$= 2\left[\arcsin\left(\frac{x}{2}\right) - \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2}\right] + C$$

$$= \boxed{2\arcsin\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + C}$$

Extra Credit (5 points)

(a) Use integration by parts to prove the reduction formula $\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$.

$\int (\ln x)^n \, dx$ $\quad u = (\ln x)^n \quad dv = dx$

$du = n(\ln x)^{n-1} \quad v = x$

$= x(\ln x)^n - n \int x(\ln x)^{n-1} \, dx$

(b) Use the reduction formula to evaluate $\int (\ln x)^2 \, dx$.

$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int (\ln x)^1 dx$

$= x(\ln x)^2 - 2\left(x(\ln x)^1 - \int dx\right)$

$= x(\ln x)^2 - 2x\ln x + 2x + C$

$$\sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)) \qquad \sin(ax)\cos(bx) = \tfrac{1}{2}(\sin((a-b)x) + \sin((a+b)x))$$

$$\cos^2(x) = \tfrac{1}{2}(1 + \cos(2x)) \qquad \sin(ax)\sin(bx) = \tfrac{1}{2}(\cos((a-b)x) - \cos((a+b)x))$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(ax)\cos(bx) = \tfrac{1}{2}(\cos((a-b)x) + \cos((a+b)x))$$