Math F252 — Midterm I — Fall 2023

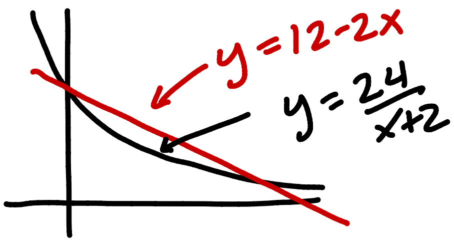
Name: Solutions

Rules:

You have 90 minutes to complete this midterm.
Partial credit will be awarded, but you must show your work.
You may have a single handwritten 3×5 notecard.
Calculators are not allowed.
Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.
Turn off anything that might go beep during the exam.
Good luck!

Problem	Possible	Score
1	10	
2	10	
3	20	
4	12	
5	8	
6	30	
7	10	
Extra Credit	5	
Total	100	

1. (10 points) Find the area between the curves $y = \dfrac{24}{x+2}$ and $y = 12 - 2x$ on the interval $[0, 4]$. Your final answer should be a reasonably simplified number.



$$A = \int_0^4 \left[12 - 2x - \left(\frac{24}{x+2}\right)\right] dx$$

$$= 12x - x^2 - 24\ln(x+2) \Big|_0^4$$

$$= 48 - 16 - 24\ln(6) + (24\ln(2))$$

$$= 32 - 24\ln(6) + 24\ln(2)$$

$$= 32 - 24(\ln(6) - \ln(2)) = 8 - 24\ln(3)$$

2. (10 points) Find the surface area of the volume generated when the curve $y = \frac{x^3}{3}$ from $x = 0$ to $x = 2$ is revolved around the x-axis. Your final answer should be a reasonably simplified number.

$$y = \frac{1}{3}x^3, \quad y' = x^2$$

$$SA = 2\pi \int_0^2 \frac{x^3}{3} \cdot \sqrt{1 + (x^2)^2}\, dx = \frac{2\pi}{3}\int_0^2 x^3 (1+x^4)^{1/2}\, dx$$

let $u = 1 + x^4$

$du = 4x^3\, dx$

$\frac{1}{4}du = x^3\, dx$

if $x = 0$, $u = 1$

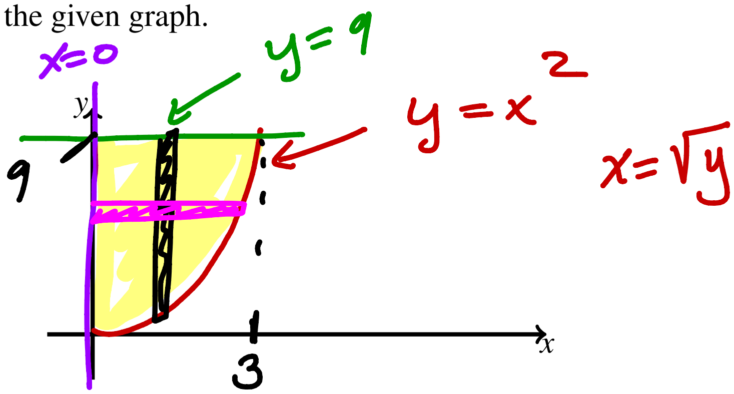
$x = 2$, $u = 17$

$$= \frac{2\pi}{3} \cdot \frac{1}{4}\int_1^{17} u^{1/2}\, du = \frac{\pi}{6} \cdot \frac{2}{3} \cdot u^{3/2} \Big|_1^{17}$$

$$= \boxed{\frac{\pi}{9}\left((17)^{3/2} - 1\right)}$$

3. (20 points) Let R be the region in the first quadrant bounded by $y = x^2$, $y = 9$, and $x = 0$.

   (a) Sketch the region R on the given graph.



   (b) For each problem below, set up – but do not evaluate – a definite integral for the volume of the solid of revolution formed by rotating the region R about the given axis using the given method. **Your final answer should be in a form that is immediately integrable without any additional algebra.**

      i. Axis of rotation: x-axis, Method: disks/washers.

vertical slice

$$V = \int_0^3 \pi\left(9^2 - (x^2)^2\right)dx = \pi\int_0^3 (81 - x^4)\,dx$$

      ii. Axis of rotation: x-axis, Method: shells.

horizontal slice

$$V = 2\pi\int_0^9 y\cdot\sqrt{y}\,dy = 2\pi\int_0^9 y^{3/2}\,dy$$

      iii. Axis of rotation: y-axis, Method: disks/washers.

horizontal slice

$$V = \pi\int_0^9 (\sqrt{y})^2\,dy = \pi\int_0^9 y\,dy$$

      iv. Axis of rotation: y-axis, Method: shells.

vertical slice

$$V = 2\pi\int_0^3 x(9 - x^2)\,dx = 2\pi\int_0^3 (9x - x^3)\,dx$$

4. (12 points) A spring has a natural length of 30 cm (or 0.3 m). It takes a force of 50 N to hold the spring at a length of 40 cm (or 0.4 m).

   (a) What is the spring constant k in Hooke's Law?

   $F = kx$ where $F = 50$ and $x = 0.1\text{m}$. So $50 = k \cdot \frac{1}{10}$ or $k = 500$

   (b) How much **work** is done to stretch the spring to a length of 50 cm (or 0.5 m)? Simplify your answer and include units.

   $$W = \int_0^{0.2} 500x\,dx = 250x^2 \Big|_0^{2/10} = 250\left(\frac{1}{25}\right) = 10\ N\cdot m$$

   (or J)

5. (8 points) Set up but do not evaluate the three integrals needed to compute the center of mass, $(\overline{x}, \overline{y})$, of the region R bounded by $y = 0$, $x = 0$, $x = 2$, and $f(x) = 3e^x$ with constant density $\rho = 1$. Then, fill in the blanks at the bottom to show how to compute the values of $(\overline{x}, \overline{y})$.

$$m = \int_0^2 3e^x\,dx$$

$$M_y = 3\int_0^2 xe^x\,dx$$

$$M_x = \frac{1}{2}\int_0^2 (3e^x)^2\,dx = \frac{1}{2}\int_0^2 9e^{2x}\,dx$$

$$\overline{x} = \frac{\boxed{M_y}}{\boxed{m}} \qquad \overline{y} = \frac{\boxed{M_x}}{\boxed{m}}$$

6. (30 points) Evaluate the following indefinite integrals. Show your work and simplify your answers.

(a) $\displaystyle\int \frac{4-2x^2}{x^3+2x}\,dx$ $= \displaystyle\int\left(\frac{2}{x} - \frac{4x}{x^2+2}\right)dx = 2\ln|x| - 2\ln(x^2+2) + C$

$x^3+2x = x(x^2+2)$

$\dfrac{4-2x^2}{x(x^2+2)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+2}$

$4-2x^2 = A(x^2+2) + x(Bx+C)$

If $x=0$: $4=2A$. So $A=2$

$0x = Cx$ So $C=0$

$-2x^2 = (A+B)x^2$ So $-2=A+B$

So $B=-4$

(b) $\displaystyle\int x^2 e^x\,dx$

$u = x^2$ $\quad dv = e^x\,dx$

$du = 2x\,dx$ $\quad v = e^x$

$= x^2e^x - 2\displaystyle\int xe^x\,dx$

$u = x$ $\quad dv = e^x\,dx$

$du = dx$ $\quad v = e^x$

$= x^2e^x - 2\left(xe^x - \displaystyle\int e^x\,dx\right)$

$= x^2e^x - 2xe^x + 2e^x + C$

$= e^x(x^2-2x+2) + C$

(c) $\int \sin^2(\theta)\cos^3(\theta)\,d\theta = \int \sin^2\theta \cdot \cos^2\theta \cdot \cos\theta \, d\theta$

Let $x = \sin(\theta)$

$dx = \cos(\theta)\,d\theta$

$= \int \sin^2\theta\,(1-\sin^2\theta)\cos\theta\,d\theta$

$= \int u^2(1-u^2)\,du = \int (u^2 - u^4)\,du$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$

$= \frac{1}{3}(\sin\theta)^3 - \frac{1}{5}(\sin(\theta))^5 + C$

(d) $\int x\sec^2(x)\,dx$

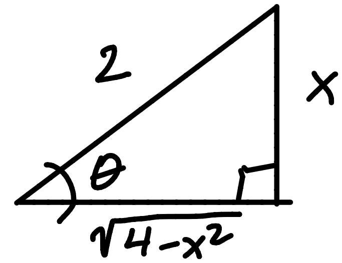
$\begin{cases} u = x & dv = \sec^2 x\,dx \\ du = dx & v = \tan x \end{cases}$

$= x\tan x - \int \tan x\,dx$

$= x\tan x - \int \frac{\sin x\,dx}{\cos x}$

$= x\tan x + \ln|\cos x| + C$

7. (10 points) Evaluate the integral $\int \frac{x^2}{\sqrt{4-x^2}}\, dx$ using trigonometric substitution. You must **fully** simplify your answer.



let $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = 2\cos\theta$

$$\int \frac{x^2\, dx}{\sqrt{4-x^2}} = \int \frac{4\sin^2\theta \cdot 2\cos\theta\, d\theta}{2\cos\theta\, d\theta} = 4\int \sin^2\theta\, d\theta$$

$$= 2\int \left(1 - \cos(2\theta)\right) d\theta = 2\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C$$

$$= 2\left(\theta - \sin(\theta)\cos(\theta)\right) + C$$

$$= 2\left[\arcsin\left(\frac{x}{2}\right) - \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2}\right] + C$$

$$= \boxed{2\arcsin\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + C}$$

Extra Credit (5 points)

(a) Use integration by parts to prove the reduction formula $\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$.

$\int (\ln x)^n \, dx$ $\quad u = (\ln x)^n \quad dv = dx$

$du = n(\ln x)^{n-1} \quad v = x$

$= x(\ln x)^n - n \int x(\ln x)^{n-1} \, dx$

(b) Use the reduction formula to evaluate $\int (\ln x)^2 \, dx$.

$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int (\ln x)^1 dx$

$= x(\ln x)^2 - 2\left(x(\ln x)^1 - \int dx\right)$

$= x(\ln x)^2 - 2x\ln x + 2x + C$

$$\sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)) \qquad \sin(ax)\cos(bx) = \tfrac{1}{2}(\sin((a-b)x) + \sin((a+b)x))$$

$$\cos^2(x) = \tfrac{1}{2}(1 + \cos(2x)) \qquad \sin(ax)\sin(bx) = \tfrac{1}{2}(\cos((a-b)x) - \cos((a+b)x))$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(ax)\cos(bx) = \tfrac{1}{2}(\cos((a-b)x) + \cos((a+b)x))$$