Math F252 **Midterm II** **Fall 2023**

Name: ____________________

Rules:

You have 90 minutes to complete this midterm.

Partial credit will be awarded, but you must show your work.

Calculators, notes and books are not allowed.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	5	
3	9	
4	10	
5	10	
6	24	
7	10	
8	10	
9	10	
Extra Credit	5	
Total	100	

1. (12 points) Compute and simplify the improper integrals, or show that they diverge. Use correct limit notation.

(a) $\int_2^\infty \frac{dx}{x(\ln(x))^2}$ $= \lim_{b\to\infty} \int_2^b \frac{(\ln x)^{-2}\,dx}{x} = \lim_{b\to\infty} -(\ln x)^{-1}\Big]_2^b$

$= \lim_{b\to\infty} \left(\frac{-1}{\ln b} + \frac{1}{\ln(2)}\right) = \frac{1}{\ln(2)}$, converges

$\frac{-1}{\ln b} \to 0$

(b) $\int_0^3 \frac{1}{x^{4/3}}\,dx$ $= \lim_{a\to 0^+}\left(\int_a^3 x^{-4/3}\,dx\right) = \lim_{a\to 0^+}\left(-3x^{-\frac{1}{3}}\Big]_a^3\right)$

$= \lim_{a\to 0^+}\left(\frac{-3}{3^{1/3}} + \frac{3}{a^{1/3}}\right) = \infty$, diverges

$\frac{3}{a^{1/3}} \to \infty$

2. (5 points) Does the series $\sum_{x=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converge or diverge? Show your work including naming any test you use. (Hint: You may use the previous problem though you don't have to.)

- Integral Test.

Since $\int_2^\infty \frac{dx}{x(\ln x)^2}$ converges, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

3. (9 points) Consider the infinite series $-\frac{1}{2\cdot 1}+\frac{1}{2\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{2\cdot 4}-\frac{1}{2\cdot 5}+\frac{1}{2\cdot 6}-\cdots$

   (a) Write the series using sigma or summation notation. (That is, write the series using $\sum$ notation.)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$$

   (b) Compute and simplify S_1, S_2, and S_3 the first three terms in the sequence partial sums of the series.

$$S_1 = \frac{-1}{2}$$

$$S_2 = \frac{-1}{2}+\frac{1}{4} = \frac{-1}{4}$$

$$S_3 = \frac{-1}{2}+\frac{1}{4}-\frac{1}{6} = \frac{-1}{4}-\frac{1}{6}$$

$$= \frac{-6-4}{24} = \frac{-10}{24} = \frac{-5}{12}$$

4. (10 points) Consider the infinite series $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{10^n}$. $= \sum_{n=0}^{\infty} (-3)\left(\frac{-3}{10}\right)^n$

   (a) Explain why the series converges.

It's geometric with $r = \frac{-3}{10}$

and $\left|\frac{-3}{10}\right| < 1$

why r -1

   (b) Determine the sum of the series. Write you answer as a simplified fraction.

$$\frac{a}{1-r} = \frac{-3}{1-\left(\frac{-3}{10}\right)} = \frac{-3}{1+\frac{3}{10}} = \frac{-3}{\frac{13}{10}} = \frac{-30}{13}$$

0

5. (10 points) Show that the series $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{2n+1}}$ is conditionally convergent.

Note that you must show that the series converges **and** that it is not absolutely convergent.

A complete answer will include (i) the name of the test(s) you are using, (ii) a clear application of the test (or tests), and (iii) an explicit explanation of what conclusion(s) you are drawing.

Show series converges.

Alternating series test with $b_n = \frac{1}{\sqrt{2n+1}}$

- $b_{n+1} = \frac{1}{\sqrt{2n+3}} < \frac{1}{\sqrt{2n+1}} = b_n$. So b_n's decreasing
- $\lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0$

So $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2n+1}}$ converges

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Show series is not absolutely convergent

Use limit comparison test. Compare $\sum_{n=0}^{\infty} \frac{1}{\sqrt{2n+1}}$ to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a divergent p-series

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{2n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{2n+1}} = \sqrt{\tfrac{1}{2}} < 1$$

So $\sum_{n=0}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{2n+1}} \right|$ diverges.

6. (24 points) Do the following series converge or diverge? Show your work, including naming any test you use.

(a) $\sum_{n=0}^{\infty} \frac{2n-1}{5n+1}$ $\boxed{\text{diverges}}$

Divergence Test

$$\lim_{n\to\infty} \frac{2n-1}{5n+1} = \frac{2}{5} \neq 0$$

(b) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converges

Limit comparison test

Compare to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, a convergent p-series

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\ln(n)}{n^2} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{\ln(n)}{n^{1/2}}$$

$$\stackrel{H}{=} \lim_{n\to\infty} \frac{1/n}{\frac{1}{2}n^{-1/2}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converges

(c) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^{5/3}}$ absolutely convergent

(direct) comparison test

$0 \leq \frac{|\sin(n)|}{n^{5/3}} \leq \frac{1}{n^{5/3}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{5/3}}$ is a convergent

p-series.

−1 no abs valus

(d) $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$ convergent

root test

$\lim_{n\to\infty} \sqrt[n]{\left|\frac{\ln(n)^n}{n^n}\right|} = \lim_{n\to\infty} \frac{\ln(n)}{n} = 0 < 1$

7. (10 points) Use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to find power series representations centered at $a = 0$ for each function below.

(a) $g(x) = \frac{x}{1-3x}$ $= x\left(\frac{1}{1-3x}\right) = x \sum_{n=0}^{\infty} (3x)^n$

$$= \sum_{n=0}^{\infty} 3^n x^{n+1}$$

(b) $h(x) = \frac{1}{(1+x)^2}$ (Hint: Differentiate an appropriate function.)

$$g(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$g'(x) = -(1+x)^{-2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$h(x) = -g'(x) = \frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$$

8. (10 points) Write the Taylor series for $y = e^{-2x}$ centered at $a = 1$.

$f(x) = e^{-2x}$ $\quad f(1) = e^{-2}$

$f'(x) = -2e^{-2x}$ $\quad f'(1) = -2e^{-2}$

$f''(x) = (-2)^2 e^{-2x}$ $\quad f''(1) = (-2)^2 e^{-2}$

$f'''(x) = (-2)^3 e^{-2x}$ $\quad f'''(1) = (-2)^3 e^{-2}$

$$e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2)^n e^{-2}}{n!}(x-1)^n$$

9. (10 points) Find the interval of convergence of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)!}$

$$\lim_{n\to\infty} \frac{\left|\frac{(x-3)^{n+1}}{(n+2)!}\right|}{\left|\frac{(x-3)^n}{(n+1)!}\right|} = \lim_{n\to\infty} \frac{|x-3|}{n+2} = 0 < 1$$

I.O.C $(-\infty, \infty)$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n8^n}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{|x|^n}{n8^n}} = \lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n}\cdot 8} = \frac{|x|}{8} < 1. \text{ So } -8 < x < 8.$$

$x = 8$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ divergent, harmonic

I.O.C $[-8, 8)$

$x = -8$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ convergent

alt. harmonic

Extra Credit (5 points) The Taylor series for $f(x) = \sin(x)$ centered at $a = 0$ is $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.

1. Find $p_3(x)$, the 3rd Taylor polynomial, and use it to estimate $\sin(1)$.
2. Show that this estimate is within ~~0.005~~ 0.01 of the exact value.

(1.) P_3 = taylor poly of degree 3

So we want first two terms (ie $n=0$ and $n=1$)

$$P_3(x) = x - \frac{x^3}{3!}$$

$n=0$

$$P_3(1) = 1 - \frac{1}{6} = \frac{5}{6}$$

(2.) The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$ is alternating. So the error is approximated by the next b_n. In this case, b_2.

$$|R_1| \leq b_2 = \frac{1}{5!} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{120} < \frac{1}{100} = 0.01$$