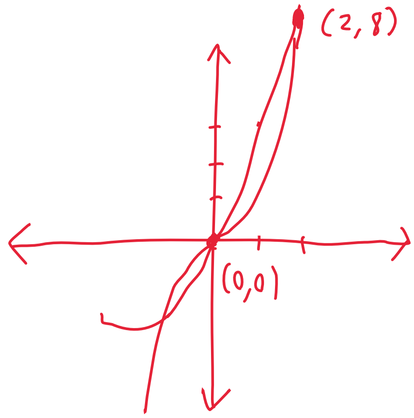
# Math F252X-001 - Fall 2024
# Exam 1

- No outside materials (e.g. books, notes, calculators, other electronic devices).
- You are allowed one 3" by 5" notecard.
- SHOW ALL WORK. Credit may not be given for answers without sufficient work.
- Illegible work will not be graded.
- Good luck!

**Print Name:** Solutions

| Page | Points | Score |
|---|---|---|
| 1 | 14 | |
| 2 | 16 | |
| 3 | 16 | |
| 4 | 10 | |
| 5 | 18 | |
| 6 | 19 | |
| 7 | 7 | |
| Total: | 100 | |

(8 pts) 1. Find the area of the region in the first quadrant bounded by the curves $y = x^2 + 2x$ and $x^3$.




$$\int_0^2 \left((x^2+2x) - x^3\right) dx = \left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\right]_0^2$$

$$= \frac{8}{3} + 4 - \frac{16}{4} = \boxed{\frac{8}{3}}$$

$$x^2 + 2x = x^3$$

$$0 = x^3 - x^2 - 2x$$

$$0 = x(x-2)(x+1)$$

$$x = -1, 0, 2$$

(6 pts) 2. Completely set up, but do not evaluate, a definite integral which gives the **arc length** of the curve $y = \sqrt{1-x^2}$ from $x = 0$ to $x = 1$.

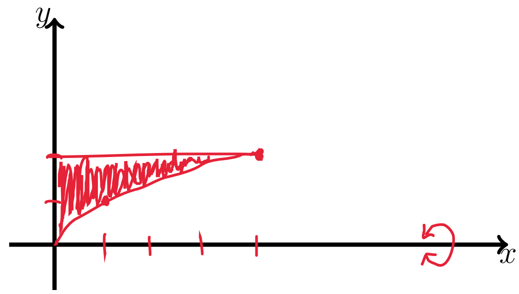
$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$\int_0^1 \sqrt{1 + \left(\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \frac{x^2}{1-x^2}}\, dx = \int_0^1 \sqrt{\frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2}}\, dx = \boxed{\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx}$$

(16 pts) 3. Let $R$ be the region bounded by the curves $y = \sqrt{x}$, $x = 0$, and $y = 2$.

(a) Sketch the region $R$. (*Hint. Double-check this part before proceeding to part (b).*)




(b) Use the **slicing (disks/washers)** method to completely set up, but not evaluate, a definite integral for the volume of the solid of revolution formed by rotating $R$ around **the $x$-axis**.

$$\int_0^4 \pi\left(2^2 - (\sqrt{x})^2\right) dx$$

(c) Use the **shell** method to completely set up, but not evaluate, a definite integral for the volume of the same solid of revolution as in part **(b)**.

$$\int_0^2 2\pi y(y^2 - 0)\, dy$$

(d) **Evaluate one** of the integrals in parts **(b)** or **(c)** to find the exact volume.

$$\pi\int_0^4 (4 - x)\, dx = \pi\left[4x - \frac{x^2}{2}\right]_0^4$$

$$= \pi(16 - 8) = \boxed{8\pi}$$

$$2\pi\int_0^2 y^3\, dy = 2\pi\left[\frac{y^4}{4}\right]_0^2$$

$$= 2\pi(4) = \boxed{8\pi}$$

(6 pts) 4. Completely set up, but do not evaluate, a definite integral for the **surface area** of the surface created when the curve $y = \sin(x)$ on the interval $x = 0$ to $x = \pi$ is rotated around **the $x$-axis**.

$$\int_0^{\pi} 2\pi \sin x \sqrt{1 + (\cos x)^2}\, dx$$

$$\frac{dy}{dx} = \cos x$$

(10 pts) 5. It takes a force of 4 Newtons to hold a spring 3 centimeters from its equilibrium.

(a) What is the spring constant $k$ in Hooke's Law (i.e. $F = kx$)?

$$4 = k(3) \Rightarrow k = \frac{4}{3}$$

(b) How much **work** is done to compress the spring 6 centimeters from its equilibrium? Simplify your answer and include units.

$$\int_0^{-6} \frac{4}{3}x\, dx = \left[\frac{2}{3}x^2\right]_0^{-6} = \frac{2}{3}(-6)^2 = \underline{24 \ \text{N}\cdot\text{cm}}$$

(10 pts) 6. Compute the center of mass $(\bar{x}, \bar{y})$ for the region in the first quadrant bounded by $y = 0$, $x = 1$ and $y = 2x^2$.

$$m = \int_0^1 2x^2\,dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3}$$

(assume $\rho = 1$)

$$m_x = \int_0^1 \frac{(2x^2)^2}{2}\,dx = \int_0^1 2x^4\,dx = \left[\frac{2}{5}x^5\right]_0^1 = \frac{2}{5}$$

$$m_y = \int_0^1 x(2x^2)\,dx = \int_0^1 2x^3\,dx = \left[\frac{1}{2}x^4\right]_0^1 = \frac{1}{2}$$

$$\bar{x} = \frac{m_y}{m} = \frac{1/2}{2/3} = \frac{3}{4} \qquad \bar{y} = \frac{m_x}{m} = \frac{2/5}{2/3} = \frac{3}{5}$$

Center of mass is $\left(\frac{3}{4}, \frac{3}{5}\right)$.

7. Compute the following integrals. Show your work.

(6 pts) (a) $\int (2^x + \frac{5}{x} + e^{-3})\,dx$

$$= \boxed{\frac{2^x}{\ln 2} + 5\ln|x| + e^{-3}x + C}$$

(6 pts) (b) $\int_0^{\pi/2} x\sin(x)\,dx$

$$= \left[-x\cos x\right]_0^{\pi/2} - \int_0^{\pi/2} -\cos x\,dx = \left[-x\cos x + \sin x\right]_0^{\pi/2}$$

Let $u = x$ $\quad v = -\cos x$

$du = dx$ $\quad dv = \sin x\,dx$

$$= \left(\left(-\frac{\pi}{2}\right)\cdot(0) + 1\right) - \left((0)(1) + 0\right)$$

$$= \boxed{1}$$

(6 pts) (c) $\int x^2 e^{(3x)}\,dx$

$$= \frac{1}{3}x^2e^{(3x)} - \frac{2}{3}\int xe^{(3x)}\,dx$$

Let $u = x^2$ $\quad v = \frac{1}{3}e^{(3x)}$

$du = 2x\,dx$ $\quad dv = e^{(3x)}dx$

Let $u = x$ $\quad v = \frac{1}{3}e^{(3x)}$

$du = dx$ $\quad dv = e^{(3x)}dx$

$$= \frac{1}{3}x^2e^{(3x)} - \frac{2}{3}\left(\frac{1}{3}xe^{(3x)} - \frac{1}{3}\int e^{(3x)}dx\right)$$

$$= \boxed{\frac{1}{3}x^2e^{(3x)} - \frac{2}{9}xe^{(3x)} + \frac{2}{27}e^{(3x)} + C}$$

(6 pts) (d) $\int \sin^2(x)\cos^5(x)\,dx$

$$= \int \sin^2 x\,(1-\sin^2 x)(1-\sin^2 x)\cos x\,dx = \int u^2(1-u^2)(1-u^2)\,du$$

Let $u = \sin x$
$du = \cos x\,dx$

$$= \int (u^2 - 2u^4 + u^6)\,du$$

$$= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C}$$

(6 pts) (e) $\int \cos(6x)\sin(2x)\,dx$

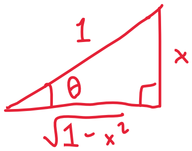
$$= \frac{1}{2}\int \left(\sin(-4x) + \sin(8x)\right)dx$$

$$= \boxed{\frac{1}{8}\cos(-4x) - \frac{1}{16}\cos(8x) + C}$$

(7 pts) (f) $\int \frac{dx}{x^2\sqrt{1-x^2}}$ Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$

$$= \int \frac{\cos\theta\,d\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta\sqrt{\cos^2\theta}} = \int \frac{d\theta}{\sin^2\theta} = \int \csc^2\theta\,d\theta$$

$$= -\cot\theta + C = \boxed{-\frac{\sqrt{1-x^2}}{x} + C}$$



(7 pts) (g) $\int \frac{x}{x^2 - x - 6}\,dx = \int \frac{x}{(x-3)(x+2)}\,dx$

$$\frac{x}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} \Rightarrow x = A(x+2) + B(x-3)$$

$$\Rightarrow \begin{array}{l} x = Ax + Bx \\ 0 = 2A - 3B \end{array} \Rightarrow A = \frac{3}{5} \quad B = \frac{2}{5}$$

$$\int \left( \frac{3/5}{x-3} + \frac{2/5}{x+2} \right) dx = \boxed{\frac{3}{5} \ln|x-3| + \frac{2}{5} \ln|x+5| + C}$$

BONUS (5 points): $\int \frac{x^2}{4 + 9x^2}\,dx$ Let $x = \frac{2}{3}\tan\theta$, $dx = \frac{2}{3}\sec^2\theta\, d\theta$

$$= \int \frac{\frac{4}{9}\tan^2\theta \, \frac{2}{3}\sec^2\theta}{4 + 9\left(\frac{4}{9}\tan^2\theta\right)}\,d\theta = \frac{8}{27} \int \frac{\tan^2\theta \sec^2\theta}{4 + 4\tan^2\theta}\,d\theta$$

$$= \frac{2}{27} \int \frac{\tan^2\theta \sec^2\theta}{\sec^2\theta}\,d\theta = \frac{2}{27} \int \tan^2\theta\,d\theta = \frac{2}{27} \int (\sec^2\theta - 1)\,d\theta$$

$$= \frac{2}{27}\tan\theta - \frac{2}{27}\theta + C = \boxed{\frac{1}{9}x - \frac{2}{27}\arctan\left(\frac{3}{2}x\right) + C}$$

You may find the following trigonometric formulas useful.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$