Math F252X-901 - Fall 2024
Exam 2

- No outside materials (e.g. books, notes, calculators, other electronic devices).
- SHOW ALL WORK. Credit may not be given for answers without sufficient work. Cite any tests/theorems you use.
- Illegible work will not be graded.

Print Name: Solutions

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Total:	100	

(6 pts) 1. Evaluate the integral $\int_2^\infty \frac{1}{x(\ln(x))^2}\,dx$.

$$\lim_{b\to\infty} \int_2^b \frac{1}{x(\ln(x))^2}\,dx \qquad u = \ln(x), \quad du = \frac{1}{x}\,dx$$

$$\lim_{b\to\infty} \int_{\ln(2)}^b \frac{1}{u^2}\,du = \lim_{b\to\infty} -\frac{1}{u}\Big|_{\ln(2)}^b = \frac{1}{\ln(2)}$$

(6 pts) 2. Evaluate the integral $\int_0^3 \frac{1}{x-1}\,dx$. $= \lim_{b\to 1^-} \int_0^b \frac{1}{x-1}\,dx + \lim_{a\to 1^+} \int_a^3 \frac{1}{x-1}\,dx$

$$= \lim_{b\to 1^-} \ln|x-1| \Big|_0^b + \lim_{a\to 1^+} \ln|x-1| \Big|_a^3$$

Both of these limits fail to exist, so the integral diverges.

(6 pts) 3. Find a closed form (explicit formula) for the sequence given by the recurrence relation $a_1 = 6000$, $a_{n+1} = \frac{a_n}{10}$.

$$a_n = 6000\left(\frac{1}{10}\right)^{n-1}$$

or

$$a_n = 6000(10)^{1-n}$$

(There are other correct answers)

(4 pts) 4. Find the first three partial sums for the series $\sum_{k=1}^{\infty}(-1)^k(2^{k-1})$.

$S_1 =$ -1

$S_2 =$ $-1+2=1$

$S_3 =$ $-1+2-4=-3$

5. Determine whether the following series converge or diverge. If a series converges, find its sum/limit.

(6 pts) (a) $\sum_{n=1}^{\infty} 3^{1/n} - 3^{1/(n+1)}$

Telescoping series

$$\sum_{n=1}^{k} 3^{1/n} - 3^{1/(n+1)} = 3 - 3^{1/(k+1)}$$

$$\text{So } \sum_{n=1}^{\infty} 3^{1/n} - 3^{1/(n+1)} = \lim_{k\to\infty}\left(3 - 3^{1/(k+1)}\right) = 2$$

(6 pts) (b) $\sum_{n=1}^{\infty} \frac{5}{2^n}$

$$= \sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^n$$

geometric series converges since $|r| = \left|\frac{1}{2}\right| < 1$

$$= \sum_{n=0}^{\infty} 5\left(\frac{1}{2}\right)^n - 5$$

$$= \frac{5}{1-\left(\frac{1}{2}\right)} - 5$$

$$= 5$$

6. Determine whether the following series coverge or diverge.

(6 pts) (a) $\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2}$

$f(x) = \frac{1}{x(\ln(x))^2}$ agrees with $\frac{1}{k(\ln(k))^2}$ on $\mathbb{N}$ and is continuous and decreasing on $[2,\infty)$. So by problem #1 and the integral test, the series converges.

(6 pts) (b) $\sum_{k=5}^{\infty} \frac{2k}{k^2 - 4k}$

Note that $0 \leq \frac{2k}{k^2-4k}$ on $[5,\infty)$ and $0 \leq \frac{1}{k}$ on $[5,\infty)$

$$\lim_{k\to\infty} \frac{2k}{k^2-4k} \cdot \frac{k}{1} = \lim_{k\to\infty} \frac{2k^2}{k^2-4k} = 2$$

Moreover $\sum \frac{1}{k}$ diverges (harmonic series).

So $\sum_{k=5}^{\infty} \frac{2k}{k^2-4k}$ diverges by the limit comparison test

(6 pts) (c) $\displaystyle\sum_{k=50}^{\infty} \frac{1}{e^{1/k}}$

$$\lim_{k\to\infty} \frac{1}{e^{1/k}} = 1 \neq 0$$

So the series diverges by the divergence test.

(6 pts) (d) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$

$$\lim_{k\to\infty} \sqrt[k]{\frac{k}{2^k}} = \lim_{k\to\infty} \frac{\sqrt[k]{k}}{2}$$

$$= \frac{1}{2}$$

Since $0 \leq \frac{1}{2} < 1$, $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$ converges by the root test.

(Ratio test also works)

(6 pts) (e) $\sum_{k=0}^{\infty} \frac{5^k}{k!}$

$$\lim_{k\to\infty} \frac{5^{k+1}}{(k+1)!} \cdot \frac{k!}{5^k} = \lim_{k\to\infty} \frac{5}{k+1}$$
$$= 0$$

converges by the ratio test.

7. Determine whether the following series converge conditionally, converge absolutely, or diverge.

(6 pts) (a) $\sum_{k=0}^{\infty} \frac{(-5)^k}{k!}$

converges absolutely by 6e.

(6 pts) (b) $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$

The sequence $\frac{1}{\sqrt{k}}$ is decreasing and $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$, so by the alternating series test, $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ converges. However, $\sum_{k=0}^{\infty} \left|\frac{(-1)^{k+1}}{\sqrt{k}}\right| = \sum_{k=0}^{\infty} \frac{1}{k^{1/2}}$ which diverges (p-series with $p=\frac{1}{2}$)

So the series converges conditionally.

(6 pts) 8. Find the center, interval, and radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$.

$\lim_{n\to\infty} \sqrt[n]{\left|\frac{nx^n}{2^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\,|x|}{2} = \frac{|x|}{2}$

$\frac{|x|}{2} < 1$ when $|x| < 2$, so radius is 2

$\sum_{n=1}^{\infty} \frac{n2^n}{2^n} = \sum_{n=1}^{\infty} n$

$\sum_{n=1}^{\infty} \frac{n(-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n$

$\lim_{n\to\infty} n = \infty \neq 0$, so both series diverge by the divergence test

center: 0 radius: 2 interval: $(-2, 2)$

9. Find power series representations for the following functions. State the radius of convergence for each. (Recall that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.)

(6 pts) (a) $f(x) = \frac{1}{1+2x}$ $= \frac{1}{1-(-2x)} = \sum_{k=0}^{\infty} (-2x)^k$

(6 pts) (b) $g(x) = \frac{2}{(1+2x)^2}$. $= -f'(x)$

$= -\frac{d}{dx} \sum_{k=0}^{\infty} (-2)^k x^k$

$= \sum_{k=0}^{\infty} (-2)^{k+1} k x^{k-1}$

(6 pts) 10. Find the Maclaurin series (Taylor series centered at 0) for the function $f(x) = \sin(x)$.

n	$f^{(n)}(x)$	$\frac{f^{(n)}(0)}{n!}$
0	$\sin(x)$	0
1	$\cos(x)$	$\frac{1}{1!}$
2	$-\sin(x)$	0
3	$-\cos(x)$	$-\frac{1}{3!}$
4	$\sin(x)$	0
5	$\cos(x)$	$\frac{1}{5!}$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

BONUS: (6 points) Determine whether the series $\sum_{k=1}^{\infty} \log\left(\frac{n}{n+1}\right)$ converges or diverges.

$$\sum_{k=1}^{\infty} \log\left(\frac{n}{n+1}\right) = \sum_{k=1}^{\infty} \left(\log(n) - \log(n+1)\right)$$

$$= \lim_{k \to \infty} \log(1) - \log(k+1) \quad \text{(telescoping series)}$$

$$= -\infty \quad \text{diverges}$$