Name: SOLUTIONS

Math 252 Calculus 2 (Bueler) Wednesday, 27 April 2022

Final Exam

No book, electronics, calculator, or internet access. 125 points possible. 125 minutes maximum.

Allowed notes: 1/2 sheet of letter paper (i.e. 8.5×11 paper) allowed, with anything written on both sides.

1. Evaluate the definite and indefinite integrals:

(a) *(6 pts)* $\displaystyle\int_0^{\pi/2} \sin^3\theta\, d\theta = \int_0^{\pi/2} \sin^2\theta \sin\theta\, d\theta = \int_0^{\pi/2} (1-\cos^2\theta)\sin\theta\, d\theta$

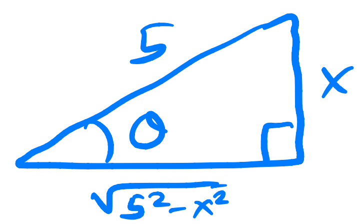
$u = \cos\theta$

$$= \int_1^0 (1-u^2)(-du) = \int_0^1 1-u^2\, du = \left[u - \frac{u^3}{3}\right]_0^1$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

(b) *(6 pts)* $\displaystyle\int \sqrt{25-x^2}\, dx = \int \sqrt{5^2 - 5^2\sin^2\theta}\; 5\cos\theta\, d\theta$

$x = 5\sin\theta$
$dx = 5\cos\theta\, d\theta$



$$= 25\int \cos^2\theta\, d\theta = \frac{25}{2}\int 1+\cos(2\theta)\, d\theta$$

$$= \frac{25}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C = \frac{25}{2}\left(\theta + \sin\theta\cos\theta\right) + C$$

$$= \frac{25}{2}\left(\arcsin\left(\frac{x}{5}\right) + \frac{x}{5}\,\frac{\sqrt{5^2-x^2}}{5}\right) + C$$

$$= \boxed{\frac{25}{2}\arcsin\left(\frac{x}{5}\right) + \frac{1}{2}x\sqrt{25-x^2} + C}$$

2. Evaluate the indefinite integrals:

(a) (*6 pts*) $\int t\,3^t\,dt =$ $\frac{t}{\ln 3}3^t - \int \frac{1}{\ln 3}3^t\,dt$

$u = t \qquad v = \frac{1}{\ln 3}3^t$

$du = dt \qquad dv = 3^t\,dt$

$= \frac{1}{\ln 3}\left(t\,3^t - \frac{1}{\ln 3}3^t\right) + C$

$= \frac{3^t}{\ln 3}\left(t - \frac{1}{\ln 3}\right) + C$

(b) (*6 pts*) $\int \frac{dx}{(x+1)(x-3)} =$

$\frac{1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

$1 = A(x-3) + B(x+1)$

$= (A+B)x + (-3A+B)$

$\left.\begin{aligned} A+B&=0 \\ -3A+B&=1 \end{aligned}\right\}$ $-4A = 1$

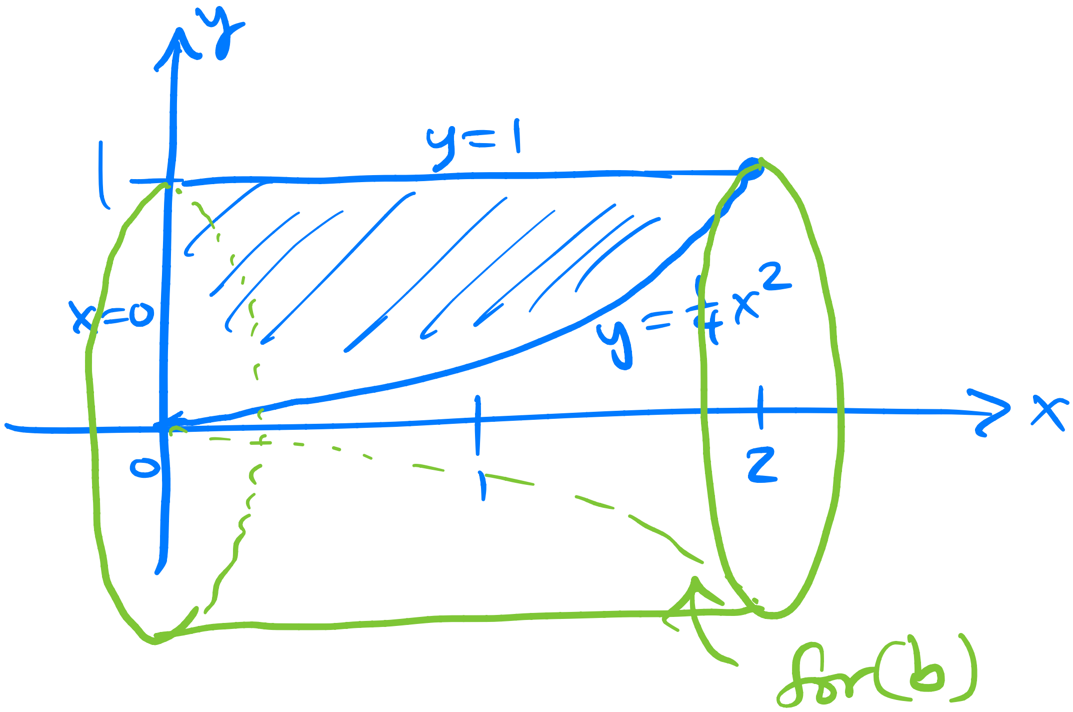
$A = -1/4$

$B = 1/4$

$= \int \frac{-1/4}{x+1} + \frac{1/4}{x-3}\,dx$

$= -\frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-3| + C$

3. **(a)** (*5 pts*) Sketch the region bounded by $y = \frac{1}{4}x^2$, the y-axis, and the line $y = 1$.



(b) (*8 pts*) Compute the volume of the solid of revolution found by rotating the region in **(a)** around the x-axis. Simplify your answer.

washers

$$V = \int_0^2 \pi\left(1^2 - \left(\tfrac{1}{4}x^2\right)^2\right)dx$$

$$= \pi\int_0^2 1 - \frac{x^4}{16}\,dx = \pi\left[x - \frac{x^5}{80}\right]_0^2$$

$$= \pi\left(2 - \frac{32}{80}\right) = \pi\left(2 - \frac{2}{5}\right) = \frac{8\pi}{5}$$

4. (*8 pts*) Compute the improper integral. Use appropriate limit notation.

$$\int_1^\infty xe^{-x^2/2}\,dx = \lim_{t\to\infty} \int_1^t x e^{-x^2/2}\,dx \quad \left\{\begin{array}{l} u = x^2/2 \\ du = x\,dx \end{array}\right.$$

$$= \lim_{t\to\infty} \int_{1/2}^{t^2/2} e^{-u}\,du = \lim_{t\to\infty} \left[-e^{-u}\right]_{1/2}^{t^2/2}$$

$$= \lim_{t\to\infty} \left[e^{-1/2} - e^{-t^2/2}\right] = e^{-1/2} - 0 = \frac{1}{\sqrt{e}}$$

5. Determine whether the following series converge or diverge. Explain your reasoning and identify any test used.

(a) (*6 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{12+n}$

limit compare to $\sum \frac{1}{\sqrt{n}}$ (diverges, $p = 1/2$)

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{12+n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}\,\sqrt{n}}{12+n} = \lim_{n\to\infty} \frac{n}{12+n} \overset{L'H}{=} 1$$

$1 \neq 0$, $1 \neq \infty$ so both series diverge

(b) (*6 pts*) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$

$$\left.\begin{array}{l} b_n = \frac{\ln n}{n} \geq 0 \\ b_n \text{ decreases} \\ \lim_{n\to\infty} b_n = 0 \end{array}\right\}$$

converges by A.S.T.

6. (*8 pts*) Find the radius and interval of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x+2)^n}{3^n\sqrt{n}}$$

ratio (or root) test

$$\rho = \lim_{n\to\infty} \frac{\frac{|x+2|^{n+1}}{3^{n+1}\sqrt{n+1}}}{\frac{|x+2|^n}{3^n\sqrt{n}}} = \lim_{n\to\infty} \frac{|x+2|\sqrt{n}}{3\sqrt{n+1}} = \frac{|x+2|}{3}\cdot 1$$

radius of conv: $R = 3$

$$\frac{|x+2|}{3} < 1 \iff -3 < x+2 < 3 \iff -5 < x < 1$$

$x=-5$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-3)^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges $(p=\frac{1}{2})$

$x=1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges AST

$\therefore$ $I = (-5, 1]$ is interval of conv.

7. (*8 pts*) Find the Taylor series for the function $f(x) = e^{2x}$ centered at the point $a = -3$. Give your answer in summation notation.

$f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 2^2 e^{2x}$

$\vdots$

$f^{(n)}(x) = 2^n e^{2x}$

$$\therefore\ c_n = \frac{f^{(n)}(-3)}{n!} = \frac{2^n e^{-6}}{n!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n e^{-6}}{n!}(x+3)^n$$

8. (*8 pts*) Find the arc length of the parametric curve defined by $x = 1 - \frac{1}{3}t^3$, $y = t^2 + 3$ on the interval $0 \le t \le 4$.

$$L = \int_0^4 \sqrt{(x')^2 + (y')^2}\, dt = \int_0^4 \sqrt{(-t^2)^2 + (2t)^2}\, dt$$

$$= \int_0^4 \sqrt{t^4 + 4t^2}\, dt = \int_0^4 t\sqrt{t^2+4}\, dt$$

$$= \int_4^{20} \sqrt{u}\, \frac{du}{2} = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_4^{20} = \frac{1}{3}\left[20^{3/2} - 4^{3/2}\right]$$

$u = t^2 + 4$

$$= \frac{8}{3}\left[5^{3/2} - 1\right] = \frac{8}{3}(5\sqrt{5} - 1)$$

9. (*6 pts*) How accurate is the approximation of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ by its partial sum S_{100}? Write a correct bound in the box and give a brief justification.

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - S_{100}\right| \le \boxed{\frac{1}{101}}$$

$$R_N = \sum_{n=1}^{\infty} \cdots - S_N$$

for alternating series, $|R_N| \le b_{N+1}$

$b_n = \frac{1}{n}$ and $N = 100$ so $b_{N+1} = \frac{1}{101}$

10. **(a)** (*5 pts*) Does the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge or diverge? Explain your reasoning and identify any test used.

root test

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2} < 1$$

$\therefore$ converges

(b) (*8 pts*) Evaluate (find the sum for) the series in **(a)** by computing $f'\left(\frac{1}{2}\right)$ where

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

$$f'(x) = \sum_{n=0}^{\infty} n x^{n-1} = (1-x)^{-2}$$

$$f'(\tfrac{1}{2}) = \sum_{n=0}^{\infty} n \left(\tfrac{1}{2}\right)^{n-1} = \left(1-\tfrac{1}{2}\right)^{-2} = \left(\tfrac{1}{2}\right)^{-2} = 2^2 = 4$$

so $$\sum_{n=0}^{\infty} \frac{n}{2^{n-1}} = 4$$

so $$\frac{1}{2}\sum_{n=0}^{\infty} \frac{n}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{n}{2^n} = 2$$

11. Consider the parametric curve $x = t + \cos t$, $y = t - \sin t$.

(a) (*6 pts*) Find the equation of the tangent line at $t = \pi$.

$$t=\pi: \quad x = \pi + (-1) = \pi - 1, \quad y = \pi - 0 = \pi$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-\cos t}{1-\sin t} \quad \therefore \quad m = \frac{1-(-1)}{1-0} = 2$$

$$\therefore \quad y - \pi = 2(x - (\pi - 1))$$

$$y = 2x - 2\pi + 2 + \pi = 2x + 2 - \pi$$

(b) (*6 pts*) Compute the second derivative $\dfrac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\left(\frac{1-\cos t}{1-\sin t}\right)'}{1-\sin t}$$

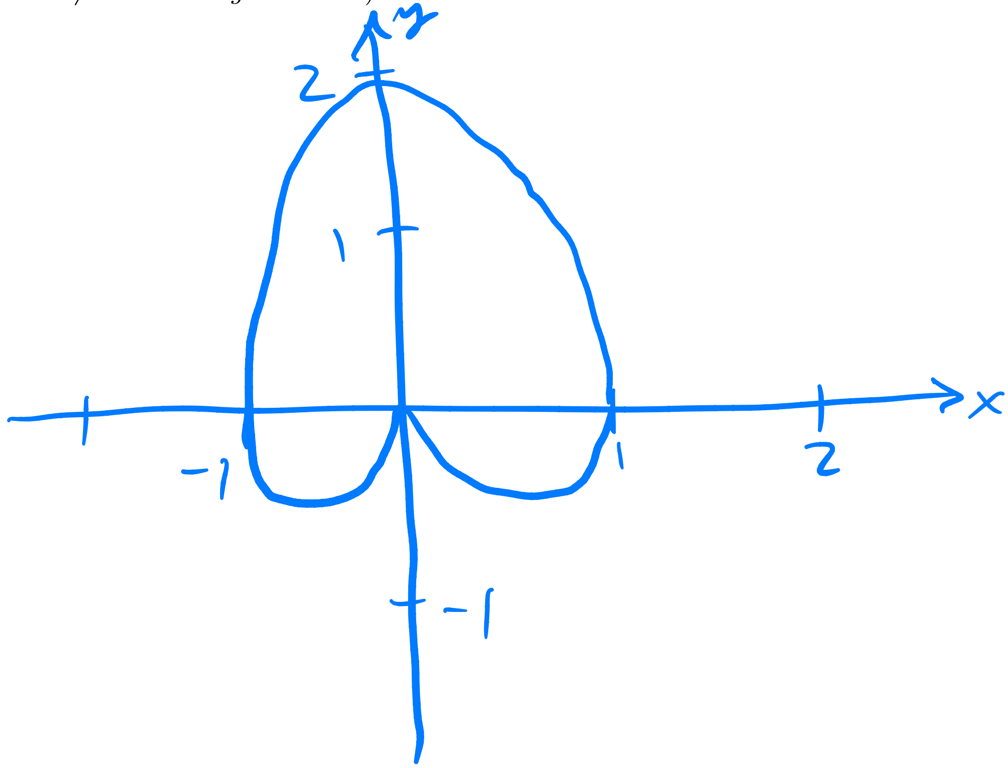
$$= \frac{\sin t(1-\sin t) - (1-\cos t)(-\cos t)}{(1-\sin t)^3}$$

$$= \frac{\sin t - \sin^2 t + \cos t - \cos^2 t}{(1-\sin t)^3}$$

$$= \frac{\sin t + \cos t - 1}{(1-\sin t)^3}$$

any of these is correct

12. **(a)** (*5 pts*) Make a careful and reasonably-large sketch of the cardiod $r = 1 + \sin\theta$. (*Label the axes and give dimensions/values along the axes.*)



θ	r
0	1
$\pi/2$	2
π	1
$3\pi/2$	0
2π	1

(b) (*8 pts*) Find the area inside the cardioid in **(a)**.

$$A = \int_0^{2\pi} \frac{1}{2}(1+\sin\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1 + 2\sin\theta + \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1 + 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos(2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin(2\theta)\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\left(\frac{3}{2}\cdot 2\pi - 2 - 0\right) - (0 - 2 - 0)\right]$$

$$= \frac{1}{2}\, 3\pi = \frac{3\pi}{2}$$

Extra Credit. (*3 pts*) These two polar curves both spiral toward the origin:

A. $r = e^{-\theta}$ on $0 \le \theta < \infty$

B. $r = \dfrac{1}{\theta}$ on $1 \le \theta < \infty$

} plot on Desmos to compare?

However, one has finite arclength and the other infinite. Which is which? Find the length of the finite one *and* show the other has infinite length.

A: $L = \int_0^\infty \sqrt{e^{-2\theta} + (e^{-\theta})^2}\, d\theta = \int_0^\infty \sqrt{2}\, e^{-\theta}\, d\theta$

$= \sqrt{2} \lim_{t\to\infty} \left[-e^{-\theta}\right]_0^t = \sqrt{2}(-0+1) = \sqrt{2}$ (finite)

B: $L = \int_1^\infty \sqrt{\left(\frac{1}{\theta}\right)^2 + \left(\frac{-1}{\theta^2}\right)^2}\, d\theta = \int_1^\infty \sqrt{\frac{1}{\theta^2} + \frac{1}{\theta^4}}\, d\theta$

$= \int_1^\infty \frac{1}{\theta}\sqrt{1 + \frac{1}{\theta^2}}\, d\theta \geq \int_1^\infty \frac{1}{\theta}\, d\theta = +\infty$

Key idea: I don't know how to do the integral on the left.

since $\lim_{t\to\infty} \ln t = \infty$

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