Name:____________________________________

Math 252 Calculus 2 (Bueler) Wednesday, 27 April 2022

Final Exam

No book, electronics, calculator, or internet access. 125 points possible. 125 minutes maximum.

Allowed notes: 1/2 sheet of letter paper (i.e. 8.5×11 paper) allowed, with anything written on both sides.

1. Evaluate the definite and indefinite integrals:

(a) (*6 pts*) $\displaystyle \int_0^{\pi/2} \sin^3\theta\, d\theta =$

(b) (*6 pts*) $\displaystyle \int \sqrt{25 - x^2}\, dx =$

2. Evaluate the indefinite integrals:

(a) (*6 pts*) $\displaystyle\int t\,3^t\,dt =$

(b) (*6 pts*) $\displaystyle\int \frac{dx}{(x+1)(x-3)} =$

3. **(a)** (*5 pts*) Sketch the region bounded by $y = \frac{1}{4}x^2$, the y-axis, and the line $y = 1$.

(b) (*8 pts*) Compute the volume of the solid of revolution found by rotating the region in **(a)** around the x-axis. Simplify your answer.

4. (*8 pts*) Compute the improper integral. Use appropriate limit notation.

$$\int_1^\infty xe^{-x^2/2}\,dx =$$

5. Determine whether the following series converge or diverge. Explain your reasoning and identify any test used.

(a) (*6 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{12+n}$

(b) (*6 pts*) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$

6. (*8 pts*) Find the radius and interval of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x+2)^n}{3^n\sqrt{n}}$$

7. (*8 pts*) Find the Taylor series for the function $f(x) = e^{2x}$ centered at the point $a = -3$. Give your answer in summation notation.

8. (*8 pts*) Find the arc length of the parametric curve defined by $x = 1 - \frac{1}{3}t^3$, $y = t^2 + 3$ on the interval $0 \le t \le 4$.

9. (*6 pts*) How accurate is the approximation of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ by its partial sum S_{100}? Write a correct bound in the box and give a brief justification.

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - S_{100} \right| \le \boxed{}$$

10. **(a)** (*5 pts*) Does the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge or diverge? Explain your reasoning and identify any test used.

(b) (*8 pts*) Evaluate (find the sum for) the series in **(a)** by computing $f'\left(\frac{1}{2}\right)$ where

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

11. Consider the parametric curve $x = t + \cos t$, $y = t - \sin t$.

(a) (*6 pts*) Find the equation of the tangent line at $t = \pi$.

(b) (*6 pts*) Compute the second derivative $\dfrac{d^2y}{dx^2}$.

12. **(a)** (*5 pts*) Make a careful and reasonably-large sketch of the cardiod $r = 1 + \sin\theta$. (*Label the axes and give dimensions/values along the axes.*)

(b) (*8 pts*) Find the area inside the cardioid in **(a)**.

Extra Credit. (*3 pts*) These two polar curves both spiral toward the origin:

A. $r = e^{-\theta}$ on $0 \le \theta < \infty$

B. $r = \dfrac{1}{\theta}$ on $1 \le \theta < \infty$

However, one has finite arclength and the other infinite. Which is which? Find the length of the finite one *and* show the other has infinite length.

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Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive. If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	This test cannot prove convergence of a series.
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\lvert r\rvert < 1$, the series converges to $a/(1-r)$. If $\lvert r\rvert \geq 1$, the series diverges.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges. If $p \leq 1$, the series diverges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.

Series or Test	Conclusions	Comments
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx.$	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0,$ then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n \to \infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert.$	If $0 \leq \rho < 1,$ the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1$ or $\rho = \infty,$ the series diverges.	
	If $\rho = 1,$ the test is inconclusive.	
Root Test For any series $\sum_{n=1}^{\infty} a_n,$ let $\rho = \lim_{n \to \infty} \sqrt[n]{\lvert a_n \rvert}.$	If $0 \leq \rho < 1,$ the series converges absolutely.	Often used for series where $\lvert a_n \rvert = b_n^n.$
	If $\rho > 1$ or $\rho = \infty,$ the series diverges.	
	If $\rho = 1,$ the test is inconclusive.	