Name: SOLUTIONS

Math 252 Calculus 2 (Bueler) Thursday, 7 April 2022

Midterm Exam 2

No book, notes, electronics, calculator, or internet access. 100 points possible. 70 minutes maximum.

1. Compute and simplify the improper integrals, or show they diverge. Use correct limit notation.

(a) (*5 pts*) $\displaystyle\int_0^\infty \frac{x^2\,dx}{1+x^3} = \lim_{t\to\infty}\int_0^t \frac{x^2\,dx}{1+x^3} = \lim_{t\to\infty}\int_1^{1+t^3}\frac{du/3}{u}$

$u = 1+x^3$

$= \lim_{t\to\infty}\frac{1}{3}\left(\ln t - \cancel{\ln 1}^{\,0}\right) = +\infty$

diverge

(b) (*5 pts*) $\displaystyle\int_0^1 \frac{dx}{\sqrt[4]{x}} = \lim_{a\to 0^-}\int_a^1 x^{-1/4}\,dx = \lim_{a\to 0^-}\left[\frac{4}{3}x^{3/4}\right]_a^1$

$= \frac{4}{3}\lim_{a\to 0^-}\left(1-a^{3/4}\right) = \frac{4}{3}(1-0) = \frac{4}{3}$

converges

2. (*5 pts*) Does the following series converge or diverge? Show your work, including naming any test you use. (*Hint.* Previous problem? Or another test?)

$$\sum_{n=0}^{\infty}\frac{n^2}{1+n^3} =$$

by integral test:

$f(x) = \dfrac{x^2}{1+x^3}$

$f(n) = a_n = \dfrac{n^2}{1+n^3}$

$\displaystyle\int_0^\infty \frac{x^2}{1+x^3}\,dx$ diverges

$\therefore$ series diverges

3. Do the following series converge or diverge? Show your work, including naming any test you use.

(a) (*5 pts*) $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{4^n}$

[root or limit comparison work]

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{n+2}{4^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n+2}}{4} = \frac{1}{4} < 1$$

$\therefore$ converges

(b) (*5 pts*) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{4^n}$

geometric with $r = -\frac{3}{4}$ so $|r| < 1$

So converge

(c) (*5 pts*) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

by AST: $b_n = \frac{1}{\ln n} \geq 0$

$\lim_{n\to\infty} b_n = 0$

b_n decreasing

$\therefore$ converges

(d) (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

ratio test

$$\rho = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)\,n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1 \quad \therefore \ \underline{\text{converges}}$$

(e) (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{(\cos n)^2}{n^2}$

comparison test

$(\cos n)^2 \le 1$

so $a_n \le \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $(p=2)$

$\therefore$ converges

4. (*5 pts*) For one of the five series in problem **3**, it is possible to compute the value of the infinite series. Which one? Explain why, and compute the value.

(b): geometric with $a=1$, $r=-3/4$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{4^n} = \frac{1}{1-(-3/4)} = \frac{1}{7/4} = \frac{4}{7}$$

5. Consider the infinite series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

(a) (*5 pts*) Write the series using sigma (Σ) notation.

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

(b) (*5 pts*) Compute and simplify S_4, the partial sum of the first four terms.

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{12-6+4-3}{12} = \boxed{\frac{7}{12}}$$

6. (*5 pts*) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converge absolutely, conditionally, or neither (diverge)? Show your work and circle one answer.

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{converges } (p=3)$$

CONVERGES ABSOLUTELY

CONVERGES CONDITIONALLY

DIVERGES

7. By any method, write a power series for the following functions. Show your work.

(a) (*5 pts*) $\dfrac{1}{1+2x}$

$$= \frac{1}{1-(-2x)} \underset{\text{geometric}}{=} \sum_{n=0}^{\infty} (-2x)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n 2^n x^n$$

(b) (*7 pts*) $\arctan x$

(*Hint.* Integrate a series derived from the geometric series.)

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \sum_{n=0}^{\infty} (-t^2)^n\,dt$$

geometric: $\dfrac{1}{1+t^2} = \dfrac{1}{1-(-t^2)}$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

8. Find the interval of convergence of the following power series.

(a) (*7 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$

root test:

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{n2^n}\right|} = \lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n}\cdot 2}$$

$$= \frac{|x|}{2} < 1 \quad \therefore \text{ converges on } -2 < x < 2$$

$\underline{x = -2:}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (AST)

$\underline{x = +2:}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic) $\therefore$ $I = [-2, 2)$

(b) (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} (n-1)!\, x^n$

$$\rho = \lim_{n\to\infty} \frac{n!\, |x|^{n+1}}{(n-1)!\, |x|^n} = \lim_{n\to\infty} n|x|$$

$$= +\infty \quad \text{if } x \neq 0$$

so

$$I = [0, 0]$$

(converges only at $x=0$)

9. (*9 pts*) Find the Taylor series of $f(x) = \dfrac{1}{x}$ at basepoint $a = 2$.

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = +2x^{-3}$

$f'''(x) = -3\cdot 2x^{-4}$

$\vdots$

$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$

$f(a) = 2^{-1}$

$\vdots$

$f^{(n)}(a) = (-1)^n n!\, 2^{-(n+1)}$

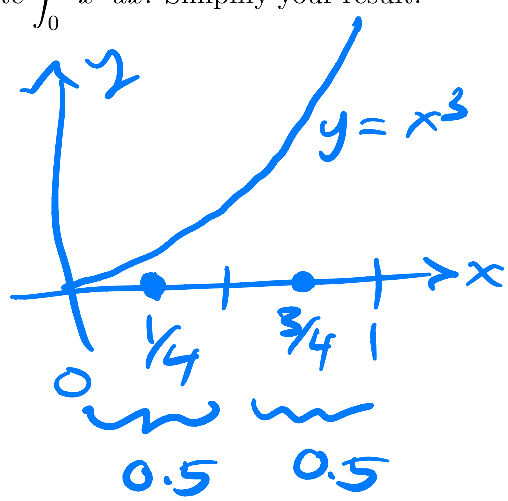
any n

$$\therefore \quad f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!\, 2^{-(n+1)}}{n!}(x-2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n$$

10. (*7 pts*) Use the midpoint rule with $n = 2$ subintervals to estimate $\displaystyle\int_0^1 x^3\,dx$. Simplify your result.

$\int_0^1 x^3\,dx \approx M_2$



$M_2 = 0.5\cdot\left(\frac{1}{4}\right)^3 + 0.5\left(\frac{3}{4}\right)^3$

$= \frac{1}{2}\left(\frac{1+27}{64}\right) = \frac{28}{128} = \frac{7}{32}$

vs. exact: $\int_0^1 x^3\,dx = 1/4$

Extra Credit. (*3 pts*) Recall the famous Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Suppose I put $x = -\frac{1}{2}$ in this power series. In fact, suppose I compute the 10th partial sum $S_{10} = \sum_{n=0}^{10} \frac{(-1/2)^n}{n!}$. I observe, using Matlab, that it gives more than 10 digits of accuracy in approximating $e^{-1/2} = \frac{1}{\sqrt{e}}$. Explain why, using a known fact about remainders.

$e^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!}$ is alternating

$\Downarrow$

$|R_{10}| = |e^{-1/2} - S_{10}| \leq b_{11} = \frac{1}{2^{11}\, 11!}$

$b_n = \frac{(1/2)^n}{n!}$

7 copies of at least 10

but $2^{11}\, 11! = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2^6$

note each "$\bigcirc$" is at least 10, and $4 \cdot 3 \cdot 2 \cdot 2^6 \geq 10^3$

so $2^{11} \cdot 11! \geq 10^{10}$ so $\boxed{b_{11} \leq 10^{-10}}$

BLANK SPACE

so 10 digit accuracy!

$4 \cdot 3 \cdot 2 \cdot 2^6$

$= 12 \cdot 128 \geq 10^3$