Math F252 **Final** Spring 2024

Name: ______________________________

Rules:

You have 2 hours to complete this midterm.

Partial credit will be awarded, but you must show your work.

Calculators and books are not allowed. You may have 1/2 of a sheet of letter paper with notes.

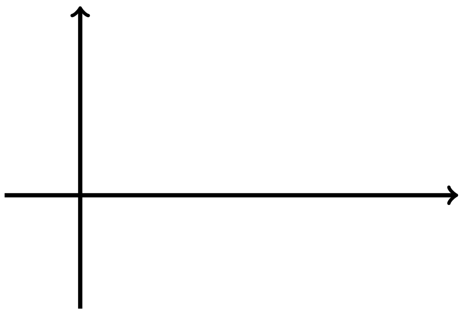
Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	6	
3	6	
4	12	
5	6	
6	6	
7	18	
8	12	
9	9	
10	15	
11	12	
12	11	
Extra Credit	*3*	
Total	125	

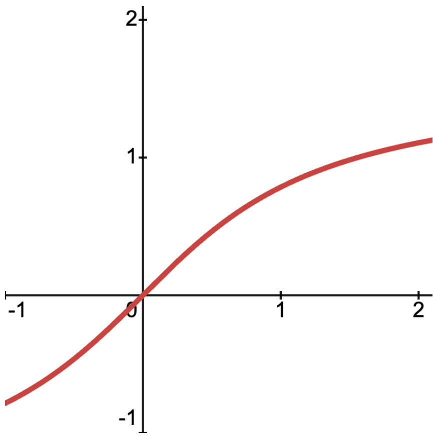
1. On the axes below, sketch the region R bounded by $y = 2\sin(x)$ and $y = 0$, between $x = 0$ and $x = \pi$.



(a) (*6 pts*) Use an integral to find the volume of the solid obtained by rotating R about the x-axis.

(b) (*6 pts*) Use an integral to find the volume of the solid obtained by rotating R about the y-axis.

2. (*6 pts*) Find the area of the region R in the plane bounded by $f(x) = \arctan(x)$, $y = 0$ and $x = 1$. The graph of arctangent is provided below. (*Hint. You will need to use a technique of integration.*)



3. (*6 pts*) Evaluate the sum $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+2}}{4^n}$.

4. Evaluate the indefinite integrals below.

(a) (*6 pts*) $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$ (*Hint. Substitute* $x = 2\sec(\theta)$.)

(b) (*6 pts*) $\displaystyle\int \sin^3(2\theta)\cos^4(2\theta)\,d\theta$

5. (*6 pts*) A spring with a relaxed length of 2 meters requires 3 Newtons force to stretch to a length of 2.1 meters. How much work would it take to stretch the spring from 2 meters to 2.3 meters?

6. (*6 pts*) Use the Integral Test to determine if the series $\displaystyle\sum_{n=0}^{\infty} \frac{2n + e^n}{(n^2 + e^n)^2}$ converges. Use correct limit notation.

7. (*6 pts each*) Do the following series converge or diverge? Show your work including **naming** any test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{3/2}}{100n^2 + 20n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{6n+5}{5n+10}\right)^n$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$

8. (*6 pts each*) For each power series below determine the **interval** of convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(3x)^n}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!(x-5)^n}{2n}$

9. Let $f(x) = \ln(x)$.

(a) (*3 pts*) Find a formula for $f^{(n)}(x)$, the nth derivative of $f(x)$.

(b) (*6 pts*) Find the Taylor series for $f(x)$ centered at $a = 1$. Your answer should be reasonably simplified.

10. Consider the curve defined by the parametric equations $x = e^t, \quad y = (t-1)^2.$

(a) (*5 pts*) Determine the slope of the curve at the point $(1, 1)$.

(b) (*5 pts*) Determine the points on the curve at which the tangent line is horizontal or vertical, or state that none exist.

(c) (*5 pts*) Set up but do not evaluate an integral for the length of the curve from $t = 1$ to $t = 2$.

11. Recall the Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) (*3 pts*) Find the Maclaurin series for $h(x) = xe^{-2x}$. Your answer should be simplified.

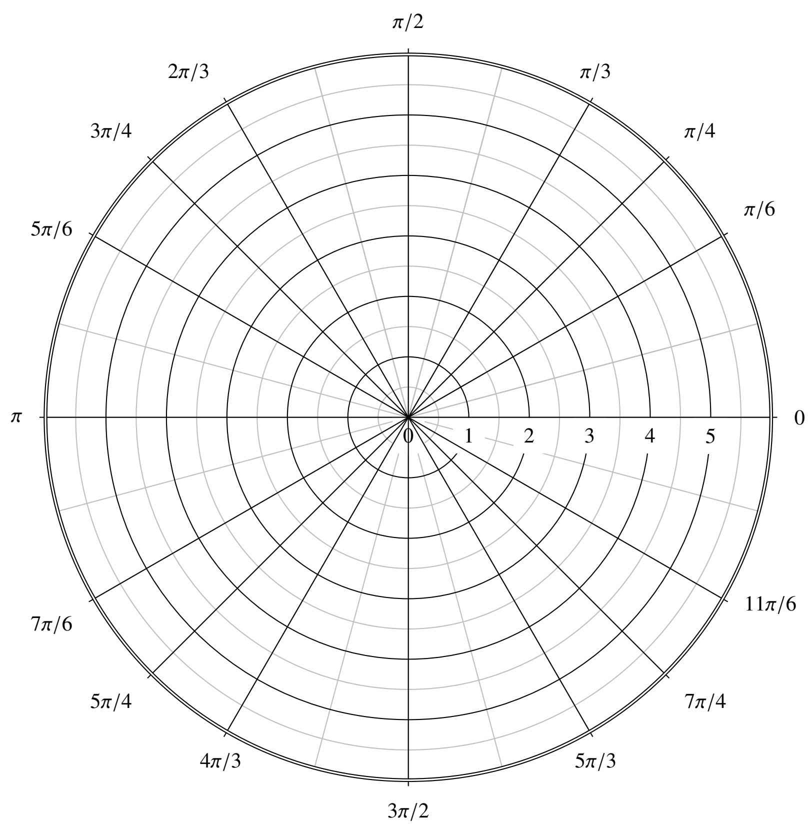
(b) (*3 pts*) Determine the value of the convergent series $\sum_{n=0}^{\infty} \frac{3^{2n}}{n!}$

(c) (*6 pts*) Find the Maclaurin series for $F(x) = \int_0^x e^{-t^2}\, dt$

12. **(a)** (*3 pts*) Convert the rectangular equation $y^2 = 5x$ to polar form.

(b) (*3 pts*) Convert the polar equation $r = \sin\theta$ to rectangular form.

(c) (*5 pts*) Sketch the polar curve $r = 2 + 2\cos\theta$.



Extra Credit. (*3 pts*) Determine the values of p for which the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges, and justify your answer. Assume $p \geq 0$.

You may find the following **trigonometric formulas** useful. Other formulas, not listed here, should be in your memory, or you can derive them from the ones here.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\|r\| < 1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
	If $\|r\| \geq 1$, the series diverges.	
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
	If $p \leq 1$, the series diverges.	
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.

Series or Test	**Conclusions**	**Comments**
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx.$	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n \to \infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert.$	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n \to \infty} \sqrt[n]{\lvert a_n \rvert}.$	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series where $\lvert a_n \rvert = b_n^n.$
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	