Math F252 Final Spring 2024

Name: SOLUTIONS

Rules:

You have 2 hours to complete this midterm.

Partial credit will be awarded, but you must show your work.

Calculators and books are not allowed. You may have 1/2 of a sheet of letter paper with notes.

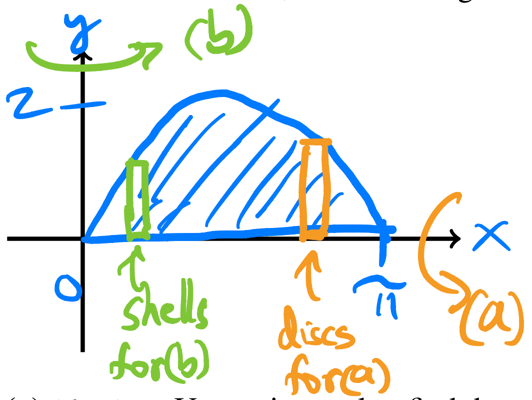
Place a box around your FINAL ANSWER to each question, or use the box provided.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	6	
3	6	
4	12	
5	6	
6	6	
7	18	
8	12	
9	9	
10	15	
11	12	
12	11	
Extra Credit	*3*	
Total	125	

1. On the axes below, sketch the region R bounded by $y = 2\sin(x)$ and $y = 0$, between $x = 0$ and $x = \pi$.



(a) (*6 pts*) Use an integral to find the volume of the solid obtained by rotating R about the x-axis.

discs

$$V = \int_0^\pi \pi(2\sin x)^2\,dx = 4\pi\int_0^\pi \tfrac{1}{2}(1-\cos 2x)\,dx$$

$$= 2\pi\left[x - \tfrac{1}{2}\sin(2x)\right]_0^\pi = 2\pi(\pi) = 2\pi^2$$

(b) (*6 pts*) Use an integral to find the volume of the solid obtained by rotating R about the y-axis.

shells:

$$V = \int_0^\pi 2\pi x \cdot 2\sin x \cdot dx = 4\pi\int_0^\pi x\sin x\,dx$$

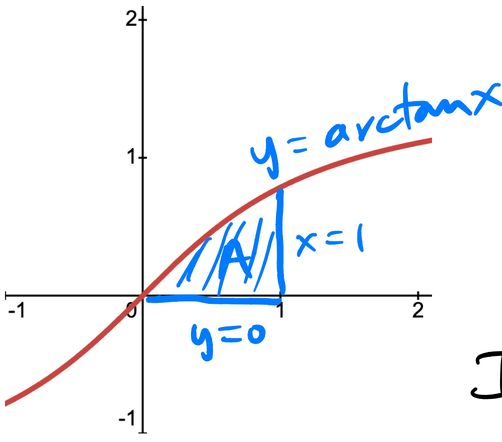
$$= 4\pi\left(x\cdot(-\cos x)\Big]_0^\pi - \int_0^\pi 1\cdot(-\cos x)\,dx\right)$$

IBP: $u = x$, $v = -\cos x$, $du = dx$, $dv = \sin x\,dx$

$$= 4\pi\left(\pi(+1) - 0 + \cancel{[\sin x]_0^\pi}^{\,0}\right) = 4\pi^2$$

(washers much more difficult . . .)

2. (*6 pts*) Find the area of the region R in the plane bounded by $f(x) = \arctan(x)$, $y = 0$ and $x = 1$. The graph of arctangent is provided below. (*Hint. You will need to use a technique of integration.*)



$$A = \int_0^1 \arctan x \, dx$$

$$= [\arctan x \cdot x]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$

IBP: $u = \arctan x$ $\quad v = x$

$du = \frac{1}{1+x^2} dx$ $\quad dv = dx$

$$= \frac{\pi}{4} \cdot 1 - 0 - \int_1^2 \frac{dw/2}{w}$$

$w = 1+x^2$

$\frac{dw}{2} = x dx$

$$= \pi/4 - \frac{1}{2}[\ln|w|]_1^2 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

3. (*6 pts*) Evaluate the sum $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+2}}{4^n}$.

geometric series: $= \frac{3^2}{1} + \frac{3^3}{4} + \frac{3^4}{4^2} + \cdots$

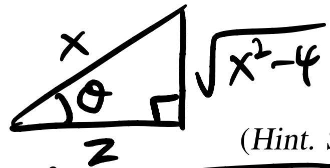
$a = 3^2 = 9, \quad r = \frac{3}{4} \qquad (|r| < 1 \checkmark)$

so

$$(\text{sum}) = \frac{9}{1 - 3/4} = \frac{9}{\frac{1}{4}} = 36$$

4. Evaluate the indefinite integrals below.

(a) *(6 pts)* $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$ (*Hint.* Substitute $x = 2\sec(\theta)$.)



$\boxed{x = 2\sec\theta} \to \sqrt{x^2-4} = 2\tan\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\sqrt{4\sec^2\theta - 4}}{\cancel{2\sec\theta}} \cdot \cancel{2\sec\theta}\tan\theta\,d\theta$$

$\boxed{1+\tan^2\theta = \sec^2\theta}$

$$= 2\int \tan\theta \cdot \tan\theta\,d\theta$$

$$= 2\int \sec^2\theta - 1\,d\theta = 2(\tan\theta - \theta) + C$$

$$= 2\left(\frac{\sqrt{x^2-4}}{2} - \arccos\left(\frac{2}{x}\right)\right) + C = \boxed{\sqrt{x^2-4} - 2\arccos\left(\frac{2}{x}\right) + C}$$

(b) *(6 pts)* $\displaystyle\int \sin^3(2\theta)\cos^4(2\theta)\,d\theta$

$\begin{cases} u = 2\theta \\ \frac{du}{2} = d\theta \end{cases}$

$$= \int \sin^2 u\cos^4 u \cdot \sin u\,\frac{du}{2}$$

$$= \frac{1}{2}\int (1-\cos^2 u)\cos^4 u \cdot \sin u\,du$$

$\begin{cases} w = \cos u \\ -dw = \sin u\,du \end{cases}$

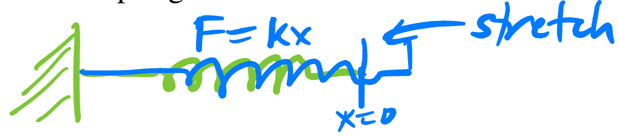
$$= -\frac{1}{2}\int (1-w^2)w^4\,dw$$

$$= -\frac{1}{2}\left[\frac{w^5}{5} - \frac{w^7}{7}\right] + C = \frac{1}{2}\left(\frac{\cos^7 u}{7} - \frac{\cos^5 u}{5}\right) + C$$

$$= \boxed{\frac{\cos^7(2\theta)}{14} - \frac{\cos^5(2\theta)}{10} + C}$$

5. (*6 pts*) A spring with a relaxed length of 2 meters requires 3 Newtons force to stretch to a length of 2.1 meters. How much work would it take to stretch the spring from 2 meters to 2.3 meters?

$F = kx$



$3N = k(0.1\,m) \Rightarrow k = 30\,\frac{N}{m}$

$W = \int F(x)\,dx = \int_0^{0.3} 30x\,dx = \frac{30}{2}x^2\Big]_0^{0.3}$

$= 15\left(\frac{3}{10}\right)^2 = \frac{15\cdot 9}{100} = \frac{135}{100} = 1.35\,J$

6. (*6 pts*) Use the Integral Test to determine if the series $\sum_{n=0}^{\infty} \frac{2n+e^n}{(n^2+e^n)^2}$ converges. Use correct limit notation.

$\int_0^{\infty} \frac{2x+e^x}{(x^2+e^x)^2}\,dx = \lim_{t\to\infty} \int_1^{t^2+e^t} \frac{du}{u^2}$

$u = x^2+e^x$
$du = (2x+e^x)\,dx$

$= \lim_{t\to\infty} -u^{-1}\Big]_1^{t^2+e^t} = \lim_{t\to\infty} \frac{-1}{t^2+e^t} + 1$

$= 0+1 = 1 \therefore$ converges

$\sum b_n$ diverges $p=\frac{1}{2}$

7. (*6 pts each*) Do the following series converge or diverge? Show your work including naming any test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n^{3/2}}{100n^2+20n}}_{=a_n}$ limit compare to: $\dfrac{n^{3/2}}{n^2} = \dfrac{1}{n^{1/2}} = b_n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^{3/2}}{100n^2+20n}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{n^2}{100n^2+20n}$$

$$\overset{LH\times 2}{=} \lim_{n\to\infty} \frac{2}{200} = \frac{1}{100} \neq 0, \neq \infty \therefore \text{ series diverges}$$

(b) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{6n+5}{5n+10}\right)^n$

root test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{6n+5}{5n+10}\right)^n\right|} = \lim_{n\to\infty} \frac{6n+5}{5n+10} = \frac{6}{5} = \rho$

$\rho > 1 \therefore$ diverges

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$

alternating series test:

$b_n = \dfrac{1}{\sqrt{2n+1}} \geq 0$

$\displaystyle\lim_{n\to\infty} b_n = 0$

$\dfrac{1}{\sqrt{2n+3}} = b_{n+1} \leq b_n = \dfrac{1}{\sqrt{2n+1}}$

converges

8. (*6 pts each*) For each power series below determine the **interval** of convergence.

root also works

(a) $\sum_{n=0}^{\infty} \frac{(3x)^n}{n^2}$

ratio test: $\lim_{n\to\infty} \frac{\frac{|3x|^{n+1}}{(n+1)^2}}{\frac{|3x|^n}{n^2}} = \lim_{n\to\infty} \frac{|3x|^{\cancel{n+1}}\, n^2}{\cancel{|3x|^n}\,(n+1)^2}$

$= |3x| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = |3x|\cdot 1 < 1 \therefore -1<3x<1$

$-\frac{1}{3}<x<\frac{1}{3}$

$x=-\frac{1}{3}$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$ converges $(p=2)$

$x=\frac{1}{3}$: $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges $(p=2)$

$[-1/3, 1/3]$

(b) $\sum_{n=0}^{\infty} \frac{(n-1)!(x-5)^n}{2n}$

ratio test: $\lim_{n\to\infty} \frac{\frac{n!\,|x-5|^{n+1}}{2(n+1)}}{\frac{(n-1)!\,|x-5|^n}{2n}}$

$= \lim_{n\to\infty} \frac{n\cancel{!}\,|x-5|^{\cancel{n}+1}\,\cancel{2}n}{\cancel{(n-1)!}\,\cancel{|x-5|^n}\,\cancel{2}(n+1)}$

$= \lim_{n\to\infty} \frac{n^2|x-5|}{n+1} = |x-5|(+\infty) = +\infty$ if $x\neq 5$

$[5,5]=\{5\}$

9. Let $f(x) = \ln(x)$.

(a) (*3 pts*) Find a formula for $f^{(n)}(x)$, the nth derivative of $f(x)$.

$f^{(0)} = \ln x$

$f^{(1)} = \frac{1}{x} = x^{-1}$

$f^{(2)} = -x^{-2}$

$f^{(3)} = +2x^{-3}$

$f^{(4)} = -3 \cdot 2 x^{-4}$

$\vdots$

$$\boxed{f^{(n)}(x) = (-1)^{n-1}(n-1)!\, x^{-n}}$$

so: $f^{(n)}(1) = (-1)^{n-1}(n-1)!$

(b) (*6 pts*) Find the Taylor series for $f(x)$ centered at $a = 1$. Your answer should be reasonably simplified.

$$\ln x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n$$

$\ln 1 = 0$

$$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(x-1)^{n+1}$$

also

10. Consider the curve defined by the parametric equations $x = e^t, \quad y = (t-1)^2.$

(a) (*5 pts*) Determine the slope of the curve at the point (1, 1). $\leftarrow$ $t=0$ gives $(x,y) = (1,1)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2(t-1)}{e^t}$$

$$\therefore \quad m = \left.\frac{dy}{dx}\right|_{t=0} = \frac{2(0-1)}{e^0} = \frac{-2}{1} = -2$$

(b) (*5 pts*) Determine the points on the curve at which the tangent line is horizontal or vertical, or state that none exist.

$$\frac{dy}{dx} = \frac{2(t-1)}{e^t} = 0 \qquad t=1 \Rightarrow (e, 0)$$ $\leftarrow$ hor. tangent

$\frac{dy}{dx}$ undefined is impossible none $\leftarrow$ vert. tangent

$e^t \neq 0$

(c) (*5 pts*) Set up but do not evaluate an integral for the length of the curve from $t = 1$ to $t = 2$.

$$L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{e^{2t} + 4(t-1)^2}\, dt$$

11. Recall the Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) (3 *pts*) Find the Maclaurin series for $h(x) = xe^{-2x}$. Your answer should be simplified.

$$h(x) = x\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{n+1}}{n!}$$

(b) (3 *pts*) Determine the value of the convergent series $\sum_{n=0}^{\infty} \frac{3^{2n}}{n!}$

$$\sum_{n=0}^{\infty} \frac{(3^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(9)^n}{n!} = e^9$$

(c) (6 *pts*) Find the Maclaurin series for $F(x) = \int_0^x e^{-t^2}\,dt$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$F(x) = \int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}$$

12. **(a)** (*3 pts*) Convert the rectangular equation $y^2 = 5x$ to polar form.

$$r^2 \sin^2\theta = 5r\cos\theta$$

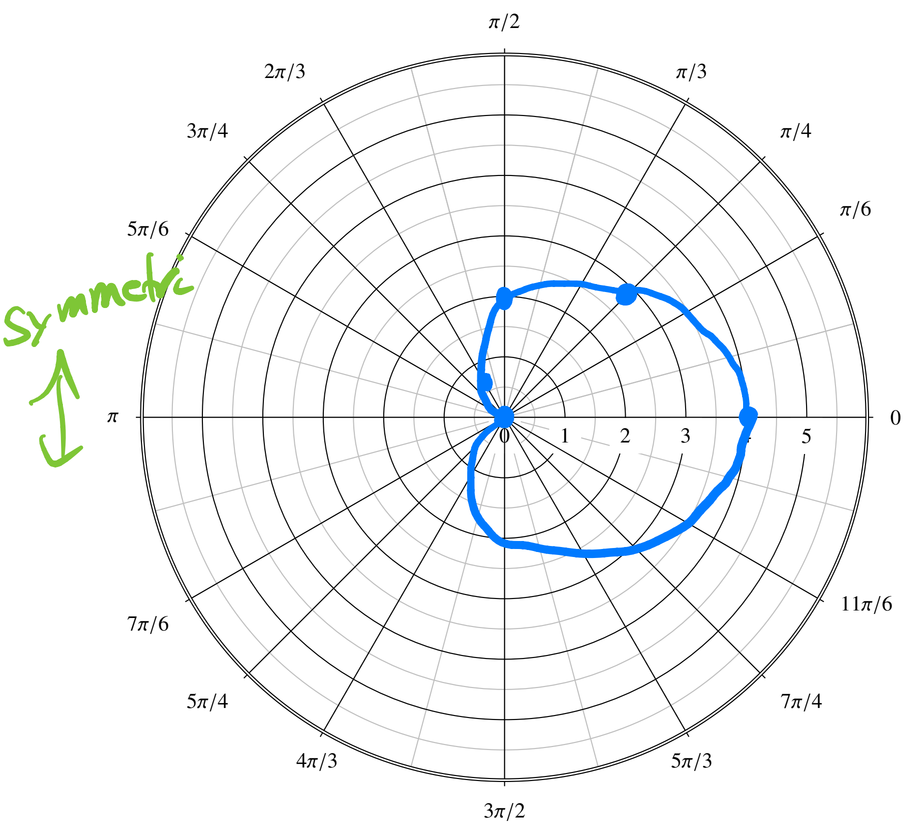
or: $r = \dfrac{5\cos\theta}{\sin^2\theta}$ etc.

(b) (*3 pts*) Convert the polar equation $r = \sin\theta$ to rectangular form.

$$r^2 = r\sin\theta$$

$$x^2 + y^2 = y$$

(c) (*5 pts*) Sketch the polar curve $r = 2 + 2\cos\theta$.



θ	r
0	4
$\pi/4$	$2+2\cdot\frac{1}{\sqrt{2}} \approx 3.4$
$\pi/2$	2
$3\pi/4$	$2-2\cdot\frac{1}{\sqrt{2}} \approx 0.6$
π	0

(a cardioid)

Extra Credit. (*3 pts*) Determine the values of p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges, and justify your answer. Assume $p \geq 0$.

apply integral test:

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^p}$$

$\leftarrow$ $u = \ln x$, $du = \frac{1}{x}$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^p}$$

$$= \lim_{t\to\infty} \begin{cases} \frac{(\ln t)^{-p+1} - (\ln 2)^{-p+1}}{-p+1}, & p \neq 1 \\ \ln(\ln t) - \ln(\ln 2), & p = 1 \end{cases}$$

$$= \begin{cases} \infty, \\ \frac{(\ln 2)^{1-p}}{p-1}, \end{cases}$$

$p \leq 1$: diverges

$p > 1$: converges

You may find the following **trigonometric formulas** useful. Other formulas, not listed here, should be in your memory, or you can derive them from the ones here.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$