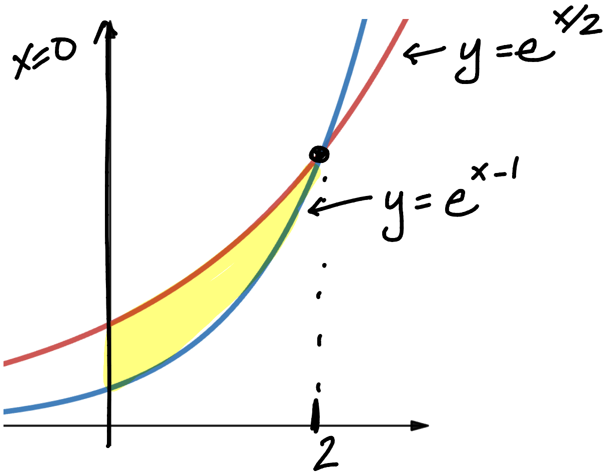
1. (a) (3 pts) Use the graphs below to shade the region bounded by $y = e^{x/2}$, $y = e^{x-1}$ and $x = 0$.



find pt. of intersection:

$e^{x/2} = e^{x-1}$ if

$\frac{x}{2} = x-1$

$x = 2x-2$

$2 = x$

(b) (7 pts) Determine the area of this region using an appropriate integral.

$$A = \int_0^2 \left(e^{x/2} - e^{x-1}\right) dx = 2e^{x/2} - e^{x-1}\Big]_0^2$$

$$= \left(2e^{2/2} - e^{2-1}\right) - \left(2e^{0/2} - e^{0-1}\right)$$

$$= 2e - e - 2 + e^{-1}$$

$$\boxed{= e + \frac{1}{e} - 2}$$

2. (5 pts) Set up but do not evaluate an integral computing the arc length of the curve $y = \tan(x^2)$ between $x = -\pi/4$ to $x = \pi/4$.

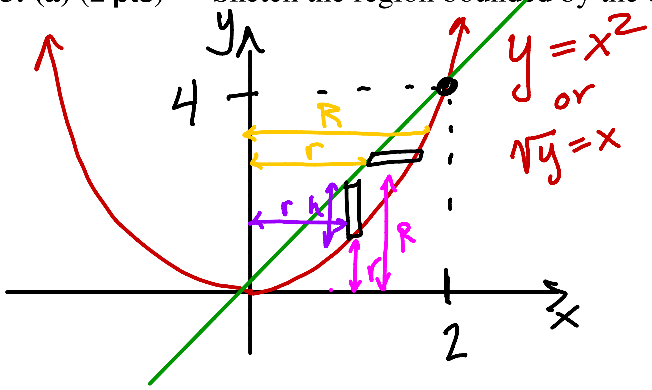
$$L = \int_\alpha^\beta \sqrt{1 + (f'(x))^2}\, dx = \boxed{\int_{-\pi/2}^{\pi/2} \sqrt{1 + 4x^2 \sec^4(x^2)}\, dx}$$

$y = \tan(x^2)$

$y' = \sec^2(x^2)(2x) = 2x \sec^2 x^2$. So $(y')^2 = 4x^2 \sec^4(x^2)$

$y = 2x$ or $x = \frac{1}{2}y$

3. **(a)** (**2 pts**) Sketch the region bounded by the curves $y = x^2$ and $y = 2x$.



pt of intersection:

$$x^2 = 2x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0,\ x = 2$$

(b) (**6 pts**) Use an integral to compute the volume of the solid found by rotating the region in part **a.** around the x-axis.

$$V = \pi \int_\alpha^\beta (R^2 - r^2)\,dx = \pi \int_0^2 \left((2x)^2 - (x^2)^2\right) dx$$

$$= \pi \int_0^2 (4x^2 - x^4)\,dx = \pi \left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \pi\left(\frac{32}{3} - \frac{32}{5}\right)$$

$$= \pi\left(\frac{2 \cdot 32}{15}\right) = \boxed{\frac{64\pi}{15}}$$

(c) (**4 pts**) Use the **shell method** to set up an integral to calculate the volume of the solid obtained by rotating the region in part **a.** around the y-axis. You do not need to evaluate the integral.

$$V = 2\pi \int_\alpha^\beta r h\,dx = 2\pi \int_0^2 x \cdot (2x - x^2)\,dx = \boxed{2\pi \int_0^2 (2x^2 - x^3)\,dx}$$

(d) (**4 pts**) Use the **slicing method** (disks/washers) to set up an integral to calculate the volume of the solid obtained by rotating the region in part **a.** around the y-axis. You do not need to evaluate the integral.

$$V = \pi \int_\alpha^\beta (R^2 - r^2)\,dy = \pi \int_0^4 (\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 dy = \boxed{\pi \int_0^4 \left(y - \frac{1}{4}y^2\right) dy}$$

4. (**10 pts**) A 3-meter long whip antenna has linear density $\rho(x) = 5 - \frac{1}{x+1}$ grams per centimeter (starting at $x = 0$). Determine the mass of the antenna. Include units.

$$\text{mass} = \int_\alpha^\beta \rho(x)\,dx = \int_0^{300}\left(5 - \frac{1}{x+1}\right)dx = 5x - \ln(x+1)\Big]_0^{300}$$

3 meters = 300 cm.

$$= \left(5\cdot 300 - \ln(300+1)\right) - \left(5\cdot 0 - \ln(0+1)\right)$$

$$\boxed{= 1500 - \ln(301) \text{ grams}}$$

5. (**10 pts**) A 1-meter spring requires 20 J to compress the spring to a length of 0.9 meters. How much work would it take to compress the spring from 1 meter to 0.8 meters?

① Find k.

$$W = 20J = \int_0^{0.1} kx\,dx = \frac{k}{2}x^2\Big]_0^{0.1} = \frac{k}{200}$$

So $4000 = k$.

② Find W.

$$W = \int_0^{0.2} 4000x\,dx = 2000x^2\Big|_0^{0.2} = \frac{2000}{25} = \boxed{80\text{ J}}$$

6. Evaluate the definite integrals. Simplify your answers

(a) (**7 pts**) $\displaystyle\int_0^{\pi/4} \tan\theta \, d\theta$

$$= \int_0^{\pi/4} \frac{\sin\theta}{\cos\theta}\, d\theta = -\ln(\cos\theta)\Big]_0^{\pi/4} = -\ln(\cos(\pi/4)) + \ln(\cos(0))$$

$$= -\ln(\sqrt{2}/2) + \ln(1)$$

$$= \underline{-\ln(\sqrt{2}/2)} = \underline{\underline{\ln(\sqrt{2})}}$$

(b) (**7 pts**) $\displaystyle\int_0^{2} xe^{3x}\, dx$

IBP
$u = x \quad dv = e^{3x}dx$
$du = dx \quad v = \frac{1}{3}e^{3x}$

$$= x\cdot\frac{1}{3}e^{3x}\Big]_0^2 - \frac{1}{3}\int_0^2 e^{3x}\,dx$$

$$= \left(\frac{2}{3}e^{6} - 0\right) - \left[\frac{1}{9}e^{3x}\right]_0^2$$

$$= \frac{2}{3}e^{6} - \left(\frac{1}{9}e^{6} - \frac{1}{9}e^{0}\right) = \boxed{\frac{5}{9}e^{6} + \frac{1}{9}}$$

7. Evaluate the indefinite integrals.

(b) (**6 pts**) $\displaystyle\int \sin^3(4x)\cos^2(4x)\,dx$

$= \int \sin^2(4x)\cos^2(4x)\sin(4x)\,dx$

$= \int (1-\cos^2(4x))\cos^2(4x)\,(\sin(4x)\,dx)$

let $u = \cos(4x)$

$du = -4\sin(4x)\,dx$

$-\frac{1}{4}du = \sin(4x)\,dx$

$= -\frac{1}{4}\int (1-u^2)u^2\,du = -\frac{1}{4}\int (u^2-u^4)\,du = -\frac{1}{4}\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C$

$$\boxed{-\frac{1}{12}\cos^3(4x) + \frac{1}{20}\cos^3(4x) + C}$$

(a) (**6 pts**) $\displaystyle\int \sec^4(x)\,dx$

$= \int \sec^2 x\,(\sec^2 x\,dx)$

$= \int (1+\tan^2 x)\sec^2 x\,dx$

let $u = \tan x$

$du = \sec^2 x\,dx$

$= \int (1+u^2)\,du = u + \frac{1}{3}u^3 + C$

$$\boxed{= \tan(x) + \frac{1}{3}\tan^3(x) + C}$$

(c) (**6 pts**) $\int \arcsin(x)\,dx$

IBP

$u = \arcsin(x)$ $\quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$ $\quad V = x$

$$= x\arcsin(x) - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

let $u = 1-x^2$, $du = -2x\,dx$

$\frac{1}{2}du = -x\,dx$

$$= x\arcsin(x) + \frac{1}{2}\int u^{-1/2}\,du$$

$$= x\arcsin(x) + u^{\frac{1}{2}} + C$$

$$\boxed{= x\arcsin(x) + \sqrt{1-x^2} + C}$$

(c) (**6 pts**) $\int \frac{2}{(2x+1)(2x-3)}\,dx$

partial fractions

$$\frac{2}{(2x+1)(2x-3)} = \frac{A}{2x+1} + \frac{B}{2x-3} \quad \text{or}$$

$$2 = A(2x-3) + B(2x+1)$$

if $x = -\frac{1}{2}$: $2 = A(-1-3) = -4A$

$A = -\frac{1}{2}$

if $x = \frac{3}{2}$: $2 = B\left(2\left(\frac{3}{2}\right)+1\right) = B(4)$

$B = \frac{1}{2}$

$$= \int\left(\frac{-1/2}{2x+1} + \frac{1/2}{2x-3}\right)dx$$

$$\boxed{= -\frac{1}{4}\ln|2x+1| + \frac{1}{4}\ln|2x-3| + C}$$

$$= \ln\left(\left|\frac{2x-3}{2x+1}\right|^{1/4}\right) + C$$

8. (**10 pts**) Use the method of Trigonometric Substitution to evaluate the integral $\int \frac{dx}{(4+x^2)^2}$. Your final answer must be simplified and written in terms of x.

Let $x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$4+x^2 = 4+4\tan^2\theta$

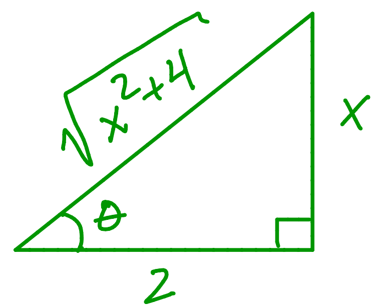
$= 4\sec^2\theta$

$$\int \frac{dx}{(4+x^2)^2}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta} = \frac{1}{8}\int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{16}\int \left(1+\cos(2\theta)\right)d\theta = \frac{1}{16}\left(\theta + \frac{1}{2}\sin(2\theta)\right)$$

$$= \frac{1}{16}\left(\theta + \sin\theta\cos\theta\right)$$



$$= \frac{1}{16}\left(\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right)$$

$\frac{x}{2} = \tan\theta$

$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

$\cos\theta = \frac{2}{\sqrt{x^2+4}}$

Extra Credit. A particle moving along a straight line has a velocity of $v(t) = te^{-t}$ after t seconds where v is measured in meters per second.

(a) (**2 pts**) How far does the particle travel from time $t = 0$ seconds to time $t = T$ seconds?

$$\text{distance travelled} = \int_0^T t e^{-t}\, dt \qquad u = t \quad dv = e^{-t} dt \qquad du = dt \quad v = -e^{-t}$$

$$= -te^{-t}\Big]_0^T + \int_0^T e^{-t}\, dt = -Te^{-T} - \Big[e^{-t}\Big]_0^T = -Te^{-T} - (e^{-T} - e^{0})$$

$$= -e^{-T}(T+1) + 1 \text{ meters.}$$

(b) (**3 pts**) Use your answer from part **a.** to determine how far the particle travels in the long-term, as $T \to \infty$.

$$\lim_{T\to\infty} \left(-e^{-T}(T+1) + 1\right) = \lim_{T\to\infty} \left(1 - \frac{T+1}{e^T}\right) = 1$$

The particle only moves 1 meter.

You may find the following **trigonometric formulas** useful. Other formulas, not listed here, should be in your memory, or you can derive them from the ones here.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$