SOLUTIONS

1. Compute and simplify the improper integrals, or show they diverge. Use correct limit notation.

(a) (*6 pts*) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}} = \lim_{a\to 0^+} \int_a^1 x^{-1/3}\,dx = \lim_{a\to 0^+} \left[\frac{x^{2/3}}{2/3}\right]_a^1$

$$= \lim_{a\to 0^+} \frac{3}{2}\left(1 - a^{2/3}\right) = \frac{3}{2}(1-0) = \frac{3}{2}$$

(b) (*6 pts*) $\displaystyle\int_1^\infty \frac{x\,dx}{1+x^2} = \lim_{t\to\infty} \int_1^t \frac{x\,dx}{1+x^2}$

$u = 1+x^2$

$\frac{du}{2} = x\,dx$

$$= \lim_{t\to\infty} \int_2^{1+t^2} \frac{du}{2u}$$

$$= \lim_{t\to\infty} \frac{1}{2}\Big[\ln u\Big]_2^{1+t^2} = \lim_{t\to\infty} \frac{1}{2}\left(\ln(1+t^2) - \ln(2)\right)$$

$\to \infty$

diverges

2. (*4 pts*) Find a formula for the general term a_n of the sequence

$\{0, 3, 8, 15, 24, 35, 48, \ldots\}$

$0 = 1-1$

$3 = 4-1 = 2^2-1$

$8 = 9-1 = 3^2-1$

$15 = 16-1 = 4^2-1$

$\vdots$

$a_n = n^2 - 1$

[or $a_n = (n-1)(n+1)$]

3. Do the following series converge or diverge? Show your work, including naming any test you use.

(a) *(6 pts)* $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2}$

limit compare to $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ ($p = 3/2$; converges)

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{n^2}}{\frac{\sqrt{n}}{n^2}} = \lim_{n\to\infty} \frac{\sqrt{n+1}\,\cancel{n^2}}{\sqrt{n}\,\cancel{n^2}} = \sqrt{\lim_{n\to\infty} \frac{n+1}{n}} = \sqrt{1} \neq 0, \infty$$

$\therefore$ converges

(b) *(6 pts)* $\displaystyle\sum_{n=1}^{\infty} \ln(n)$

$\displaystyle\lim_{n\to\infty} \ln n = +\infty$ $\therefore$ diverges by divergence test

(c) *(6 pts)* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

$b_n = \dfrac{1}{\sqrt{n+1}} \geq 0$, $\displaystyle\lim_{n\to\infty} b_n = 0$, b_n decreases

converges by alternating series test

4. Do the following series converge or diverge? Show your work, including naming any test you use.

(a) (*6 pts*) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(n+2)!}$

ratio test: $\displaystyle\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+3)!}}{\frac{2^n}{(n+2)!}} = \lim_{n\to\infty} \frac{2^{\cancel{n}+1}\,\cancel{(n+2)!}}{\cancel{2^n}\,(n+3)\cancel{(n+2)!}}$

$\displaystyle = \lim_{n\to\infty} \frac{2}{n+3} = 0 = \rho < 1 \quad \therefore$ converges

(b) (*6 pts*) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n+1}{2n+3}\right)^n$

root test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{2n+3}\right)^n} = \lim_{n\to\infty} \frac{n+1}{2n+3}$

$\displaystyle = \frac{1}{2} = \rho < 1 \quad \therefore$ converges

(c) (*6 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{(n^2)}}$

integral test: $\displaystyle\int_1^\infty x e^{-x^2}\,dx = \lim_{t\to\infty} \int_1^{t^2} e^{-u}\,\frac{du}{2}$

$\left(u = x^2,\ \frac{du}{2} = x\,dx\right)$

$\displaystyle = \lim_{t\to\infty} -\frac{1}{2}\left[e^{-u}\right]_1^{t^2} = \lim_{t\to\infty} -\frac{1}{2}\left[e^{-t^2} - e^{-1}\right] = +\frac{1}{2e} < \infty$

$\therefore$ converges

also works: root, limit comparision, ratio

5. Consider the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$

(a) (*4 pts*) Write the series using sigma ($\sum$) notation.

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

(b) (*4 pts*) Compute and simplify S_3, the partial sum of the first three terms.

$$S_3 = 1 - \frac{1}{3} + \frac{1}{5} = \frac{15-5+3}{15} = \frac{13}{15}$$

(c) (*5 pts*) Does the series converge absolutely, conditionally, or neither (diverge)? Show your work, identify any test(s) used, and circle one answer.

alternating series test: $b_n = \frac{1}{2n+1} \geq 0$, $\lim_{n\to\infty} b_n = 0$, b_n decreases

$\therefore$ series converges

take abs. vals: $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ limit compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ (diverges) ← also integral test

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0, \infty \quad \therefore \text{abs. series diverges}$$

CONVERGES ABSOLUTELY | CONVERGES CONDITIONALLY | DIVERGES

6. Use the well known geometric series $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ to find power series representations for the following functions. Show your work. (*Hint on part* **(b)**: *Use the answer from part* **(a)**.)

(a) (*6 pts*) $\frac{1}{1+x^2}$

use $r=-x^2$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\boxed{\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}}$$

(b) (*6 pts*) $\arctan x$

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\,dt = \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{2n+1}}{2n+1}\right]_0^x$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

7. (*5 pts*) Compute and simplify the value of the infinite series $\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^{n+1} = \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \cdots$

geometric with $a = \left(\frac{1}{5}\right)^2$, $r = \frac{1}{5}$:

$$\sum \cdots = \frac{a}{1-r} = \frac{(1/5)^2}{1-1/5} = \frac{1}{5^2}\cdot\frac{5}{4} = \frac{1}{20}$$

8. (*6 pts*) If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, find a *simplified* power series representation for $f'(-x^2)$.

$$f'(x) = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

note we lose the constant term

$$f'(-x^2) = \sum_{n=1}^{\infty} \frac{(-x^2)^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(n-1)!}$$

this yields full credit

[$k=n-1$]

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$$

$f'(-x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$

9. Find the **radius** and **interval** of convergence of the following power series.

(a) (*6 pts*) $\sum_{n=1}^{\infty} \frac{3^n x^n}{n!}$

ratio test: $\lim_{n\to\infty} \dfrac{\frac{3^{n+1}|x|^{n+1}}{(n+1)!}}{\frac{3^n|x|^n}{n!}} = \lim_{n\to\infty} \dfrac{\cancel{3^{n+1}}\,|x|^{\cancel{n+1}}\,\cancel{n!}}{\cancel{3^n}\,\cancel{|x|^n}\,(n+1)\cancel{n!}}$

$= \lim_{n\to\infty} \dfrac{3|x|}{n+1} = 0 = \rho < 1$ always (for all x)

$R = \infty$

interval: $(-\infty, \infty)$

(b) (*6 pts*) $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\,2^n}$

ratio test: $\lim_{n\to\infty} \dfrac{\frac{|x+1|^{n+1}}{(n+1)2^{n+1}}}{\frac{|x|^n}{n2^n}} = \lim_{n\to\infty} \dfrac{|x+1|^{\cancel{n+1}}\, n\, \cancel{2^n}}{\cancel{|x+1|^n}(n+1)\,2^{\cancel{n+1}}}$

$= \dfrac{|x+1|}{2} \lim_{n\to\infty} \dfrac{n}{n+1} = \dfrac{|x+1|}{2} = \rho < 1 \iff |x+1| < 2 \;{=R}$

$\iff -3 < x < 1$

$\underline{x=-3}$: $\sum_{n=1}^{\infty} \dfrac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ converges AST

$\underline{x=1}$: $\sum_{n=1}^{\infty} \dfrac{2^n}{n2^n} = \sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges harmonic

$R = 2$

interval: $[-3, 1)$

S ← see problem 5

Extra Credit. (*3 pts*) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ converges to $\pi/4$. Suppose you wanted to use this series to obtain an estimate of $\pi/4$ that is within 0.0001 of the actual value. Determine the fewest number of terms you would need to sum in order to obtain this level of accuracy. Explain your reasoning.

$R_N = S - S_N$

main idea!

$|R_N| \le b_{N+1}$ } for alternating series

$|R_N| \le \frac{1}{2(N+1)+1} < 10^{-4}$

$\Longleftrightarrow \quad 2N+3 > 10^4$

$N > \frac{10^4 - 3}{2} = \frac{9997}{2} = 4998.5$

$\Uparrow$

$N = 4999$

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