Math 252: Quiz 6 **20 October, 2022**

Name: SOLUTIONS

/ 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [9 points] Compute and simplify the improper integrals, or show that they diverge. Use correct limit notation.

a. $\int_0^\infty \frac{1}{4+x^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big]_0^t = \lim_{t\to\infty} \frac{1}{2}\left(\arctan\left(\frac{t}{2}\right) - 0\right)$

$= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

[$\underline{\text{or by trig. subst.}}$: $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$

gives $\lim_{t\to\infty}\int_0^{\arctan(t/2)} \frac{2\sec^2\theta\, d\theta}{4\sec^2\theta} = \cdots = \frac{\pi}{4}$]

b. $\int_0^\infty \sin x\,dx = \lim_{t\to\infty}\int_0^t \sin x\,dx = \lim_{t\to\infty}\Big[-\cos x\Big]_0^t$

$= \lim_{t\to\infty} 1 - \cos t$ d.n.e.

c. $\int_0^1 \frac{1}{\sqrt[4]{x}}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-1/4}\,dx = \lim_{t\to 0^+} \frac{4}{3}x^{3/4}\Big]_t^1$

$= \frac{4}{3}\left(1 - \lim_{t\to 0^+} t^{3/4}\right) = \frac{4}{3}(1-0) = \frac{4}{3}$

2. [4 points] Verify that $y = e^{x^2/2}$ solves the differential equation $y' = xy$.

$$y' \stackrel{?}{=} x\,y$$

$$e^{x^2/2}(x) = x\,e^{x^2/2} \quad \checkmark$$

3. [4 points] Find the general solution to the differential equation $y' = \ln x + \tan x$.

$$y(x) = \int \ln x\,dx + \int \tan x\,dx$$

$[u = \cos x$
$-du = \sin x\,dx]$

$$= x\ln x - \int x \cdot \frac{1}{x}\,dx + \int \frac{-du}{u}$$

[by parts: $u = \ln x$, $dv = dx$]

$$= x\ln x - x - \ln|\cos x| + C$$

$$= x\ln x - x + \ln|\sec x| + C$$

4. [4 points] Find the particular solution of the differential equation $y' = \dfrac{y}{x^2}$ that passes through $\left(1, \dfrac{2}{e}\right)$, given that $y = Ce^{-1/x}$ is the general solution.

$y = Ce^{-1/x}$ at $x=1$, $y = \frac{2}{e}$

$\frac{2}{e} = Ce^{-1/1} = Ce^{-1} = \frac{C}{e}$

$2 = C$

$\therefore$ $y(x) = 2e^{-1/x}$

5. [4 points] Find the area of the region in the first quadrant between the curve $y = e^{-3x}$ and the x-axis.

$A = \int_0^\infty e^{-3x}\,dx = \lim_{t\to\infty} \int_0^t e^{-3x}\,dx$

$= \lim_{t\to\infty} \left[\frac{e^{-3x}}{-3}\right]_0^t = \frac{1}{3} \lim_{t\to\infty} \left(1 - e^{-3t}\right)$

$= \frac{1}{3}(1-0) = \frac{1}{3}$

EC. [1 points] (**Extra Credit**) Suppose the curve $y = e^{-x}$, on the interval $[0, +\infty)$, is rotated around the x-axis. Set up an integral for the surface area and give a convincing argument that this integral is finite.

$$A = \int_0^\infty 2\pi e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$$

$$= 2\pi\int_0^\infty e^{-x}\sqrt{1+e^{-2x}}\,dx$$

quick way to show finite:

$e^{-2x} \le 1$ on $[0,\infty)$

$1+e^{-2x} \le 2$

$\sqrt{1+e^{-2x}} \le \sqrt{2}$

show it is 1

$$\text{so} \quad A \le \int_0^\infty 2\pi e^{-x}\sqrt{2}\,dx = 2\sqrt{2}\pi\underbrace{\int_0^\infty e^{-x}\,dx}_{} = 2\sqrt{2}\pi \approx 8.885$$

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slow way: $[u = e^{-x}]$ gives $A = \lim_{t\to\infty} -\int_1^{e^{-t}} 2\pi\sqrt{1+u^2}\,du$

then $[u = \tan\theta]$ gives $A = \lim_{t\to\infty} 2\pi\int_{\arctan(e^{-t})}^{\pi/4} \sec\theta\sec^2\theta\,d\theta$

$$= 2\pi\int_0^{\pi/4}\sec^3\theta\,d\theta = \underbrace{\ldots}_{\text{gory mess}} = 7.2118$$