Math 252: Quiz 6 **19 Oct 2023**

Name: Solutions ______ / 25

30 minutes maximum. 25 possible points. No aids (book, calculator, etc.) are permitted Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form.

1. **[10 points]** Evaluate the improper integrals below. Full points will be awarded only if the solution is written using proper notation.

   a. $\displaystyle\int_0^{\infty} \frac{2}{25+x^2}\,dx = \lim_{b\to\infty} \frac{2}{25}\int_0^b \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx = \lim_{b\to\infty} \frac{2}{5}\arctan\left(\frac{x}{5}\right)\Big]_0^b$

   $\displaystyle= \lim_{b\to\infty} \frac{2}{5}\left(\arctan\left(\frac{b}{5}\right) - \underbrace{\arctan(0)}_{0}\right) = \frac{2}{5}\cdot\frac{\pi}{2} = \frac{\pi}{5}$

   converges

   b. $\displaystyle\int_2^{6} \frac{1}{\sqrt{6-x}}\,dx = \lim_{b\to 6^-}\left(\int_2^b (6-x)^{-1/2}\,dx\right) = \lim_{b\to 6^-} -2(6-x)^{\frac{1}{2}}\Big]_2^b$

   $\displaystyle= \lim_{b\to 6^-} -2\sqrt{6-b} + 2\sqrt{6-2} = 0 + 4 = 4$

   converges

   c. $\displaystyle\int_0^{10} \frac{1}{x^{\pi}}\,dx = \lim_{a\to 0^+}\int_a^{10} x^{-\pi}\,dx = \lim_{a\to 0^+}\left(\frac{1}{1-\pi}\,x^{1-\pi}\right]_a^{10}$

   $\displaystyle= \lim_{a\to 0^+} \frac{1}{1-\pi}\left(10^{1-\pi} - a^{1-\pi}\right) = \lim_{a\to 0^+}\left(\frac{-1}{\pi-1}\right)\left(10^{1-\pi} - \frac{1}{a^{\pi-1}}\right)$

   $= +\infty.$ Diverges

2. [5 points] Find the area of the region in the first quadrant between the curve $y = e^{-4x}$ and the x-axis.

$$A = \int_0^\infty e^{-4x}\,dx = \lim_{b\to\infty} \int_0^b e^{-4x}\,dx = \lim_{b\to\infty} \left(-\frac{1}{4}e^{-4x}\right]_0^b$$

$$= \lim_{b\to\infty} -\frac{1}{4}\left(e^{-4b} - e^{0}\right) = -\frac{1}{4}(0-1) = \frac{1}{4}$$

3. [10 points] For each sequence, (i) find the first four terms (no simplification required) and (ii) determine whether the sequence converges or diverges. If it converges, find its limit.

form ∞/∞

a. $a_n = \dfrac{\ln(n^3)}{\ln(5n)}$

$a_1 = 0$

$a_2 = \ln(2^3)/\ln(10)$

$a_3 = \ln(3^3)/\ln(15)$

$a_4 = \ln(4^3)/\ln(20)$

$$\lim_{n\to\infty} \frac{\ln(n^3)}{\ln(5n)} = \lim_{n\to\infty} \frac{3\ln(n)}{\ln(5)+\ln(n)}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{3\cdot\frac{1}{n}}{\frac{1}{n}} = 3$$

converges

b. $a_n = \dfrac{100}{n!}$

$$\lim_{n\to\infty} \frac{100}{n!} = 0$$

because $n! \to \infty$ as $n \to \infty$.

$a_1 = 100$

$a_2 = 100/2$

$a_3 = 100/6$

$a_4 = 100/24$

converges

c. $a_1 = 4$, $a_{n+1} = \dfrac{1}{3}a_n$

$$\lim_{n\to\infty} 4\left(\frac{1}{3}\right)^{n-1} = \lim_{n\to\infty} \frac{4}{3^{n-1}} = 0$$

$a_1 = 4$

$a_2 = 4\cdot\frac{1}{3}$

$a_3 = 4\left(\frac{1}{3}\right)^2$

$a_4 = 4\left(\frac{1}{3}\right)^3$

$\vdots$

$a_n = 4\left(\frac{1}{3}\right)^{n-1}$

converges.