**Math 252: Quiz 7** **26 Oct 2023**

**Name:** Solutions ______ / 25

30 minutes maximum. 25 possible points. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form.

**1. [8 points]** For each series below (i) write the series using $\sum$ notation, (ii) determine whether the series converges, (iii) explain your reasoning, (iv) if the series converges, determine its sum.

**a.** $2+\frac{2}{\pi}+\frac{2}{\pi^2}+\frac{2}{\pi^3}+\frac{2}{\pi^4}+\cdots$ (i) $= \sum_{n=1}^{\infty} 2\left(\frac{1}{\pi}\right)^{n-1}$

(ii) $\sum_{n=1}^{\infty} 2\left(\frac{1}{\pi}\right)^{n-1}$ converges (iii) It is a convergent geometric series, $|r| = \frac{1}{\pi} < 1$.

(iv) $\sum_{n=1}^{\infty} 2\left(\frac{1}{\pi}\right)^{n-1} = \frac{2}{1-\frac{1}{\pi}} = \frac{2}{\frac{\pi-1}{\pi}} = \frac{2\pi}{\pi-1}$

**b.** $-\frac{4}{3}+\frac{16}{9}-\frac{64}{27}+\frac{256}{81}-\cdots = \sum_{n=1}^{\infty}\left(-\frac{4}{3}\right)^n = \sum_{n=1}^{\infty}\left(-\frac{4}{3}\right)\left(\frac{-4}{3}\right)^{n-1}$

A divergent geometric series because

$|r| = \left|-\frac{4}{3}\right| = \frac{4}{3} > 1.$

**2. [3 points]** Given the series $\sum_{n=1}^{\infty}\left(\frac{3}{n+3}-\frac{3}{n+4}\right)$.

**a**. Find $S_k$, the $k$th partial sum of the series.

$$S_k = \left(\frac{3}{4}-\frac{3}{5}\right)+\left(\frac{3}{5}-\frac{3}{6}\right)+\dots+\left(\frac{3}{k+2}-\frac{3}{k+3}\right)+\left(\frac{3}{k+3}-\frac{3}{k+4}\right)$$

$$= \frac{3}{4}-\frac{3}{k+4}$$

**b**. Use $S_k$ to determine the value of series or explain why the series diverges.

$$\lim_{k\to\infty}\left(\frac{3}{4}-\frac{3}{k+4}\right) = \frac{3}{4}$$

So $\sum_{n=1}^{\infty}\left(\frac{3}{n+3}-\frac{3}{n+4}\right)$ converges to $\frac{3}{4}$.

**3. [4 points]** Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges.

$$\int_1^{\infty} \frac{x\,dx}{e^{x^2}} = \lim_{b\to\infty} \int_1^b \frac{x\,dx}{e^{x^2}} = \lim_{b\to\infty} \left.\frac{-1}{2e^{x^2}}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{2}\left(\frac{1}{e^{b^2}} - \frac{1}{e}\right)\right] = \frac{1}{2e}$$

So $\sum_{n=1}^{\infty} n e^{-n^2}$ converges.

**4. [2 points]** State what is meant by the **harmonic series** and whether the series converges or diverges.

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

**5. [8 points]** Determine whether the series below converge or diverge. Explain your reasoning.

**a.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}} = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$

$p = \frac{4}{3} > 1$. This is a convergent p-series.

**b.** $\sum_{n=1}^{\infty} \frac{n}{\ln(n)}$

$\lim_{n\to\infty} \frac{n}{\ln n} \overset{H}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty$

So the series diverges by the Divergence Test.